THE SHINTANI DESCENT OF A CUSPIDAL REPRESENTATION OF $\mathbf{GL}_n(k_d)$

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The authors dedicate this paper to the memory of T. Shintani.

ABSTRACT. Let k be a finite field, $k_d|k$ the degree d extension of k, and $\mathrm{G}(k_d) := \mathrm{GL}_n(k_d)$ the general linear group with entries in k_d. Shintani (T. Shintani, Two Remarks on Irreducible Characters of Finite General Linear Groups, J. Math. Society of Japan 28 (1976), 396–414) showed how to associate to any generator ϕ of the Galois group $\mathrm{Gal}(k_d|k)$ a bijective mapping j_ϕ from the set of $\mathrm{Gal}(k_d|k)$-invariant characters of $\mathrm{G}(k_d)$ to the set of all irreducible characters of $\mathrm{G}(k) := \mathrm{GL}_n(k)$ which in a natural way generalizes the bijection $\chi \circ \mathrm{N}_{k_d|k} \mapsto \chi$ from the set of $\mathrm{Gal}(k_d|k)$-invariant characters of $k_d^\times = \mathrm{GL}_1(k_d)$ to the set of characters χ of $k^\times = \mathrm{GL}_1(k)$. Let Θ be a $\mathrm{Gal}(k_d|k)$-invariant cuspidal character of $G(k_d)$. Then $(d, n) = 1$ and the Green's parameter $[\chi_{dn}]$ of Θ is of the form $[\chi_n \circ \mathrm{N}_{k_{dn}|k_n}]$, where $[\chi_n]$ is a $\mathrm{Gal}(k_n|k)$-orbit of $k_n|k$-regular characters of $k_n^\times$ and $\mathrm{N}_{k_{dn}|k_n} : k_{dn} \to k_n$ is the norm mapping. In this paper we show that $j_\phi(\Theta)$ is cuspidal and that the Green's parameter of $j_\phi(\Theta)$ is $[\chi_n]$, independent of the choice of generator ϕ. These facts follow from more general assertions proved by global methods for GL_n over local fields (J. G. Arthur and L. Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Annals of Math. Study 120, Princeton U. Press, 1989). The arguments of the present paper make use of only the classical theory of representations of general linear groups over finite fields.

0. INTRODUCTION

§0.1 Some Notation.

Let $k = \mathbb{F}_q$ be a finite field of cardinality q, let $\bar{k}$ be an algebraic closure of k, and let $\mathcal{F} : x \mapsto x^q$ be the Frobenius morphism of $\bar{k}|k$. Let k_ℓ, $k \subseteq k_\ell \subset \bar{k}$, denote the fixed field of the cyclic group $\langle \mathcal{F}^\ell \rangle$ for any $\ell \geq 1$.

For any positive integer m we write $\mathrm{G}_m := \mathrm{GL}_m$, considered as an algebraic group. We fix a positive integer n and write $\mathrm{G} := \mathrm{G}_n$. For any k_ℓ-subgroup H of G_m we write $\mathrm{H}(k_\ell)$ to denote the group of k_ℓ-points of H.

For any finite group Y we write $\mathrm{Y}^\wedge$ to denote the set of irreducible unitary representations of Y and $X(\mathrm{Y})$ for the group of one-dimensional characters of Y or $\mathrm{Y}/[\mathrm{Y}, \mathrm{Y}]$.

We let $\mathcal{F}$ act on $\mathrm{G}(k_d)$ by letting it act on the coefficients of $g \in \mathrm{G}(k_d)$. Thus we may represent this action as an action of the finite cyclic group $\langle \mathcal{F} \rangle / \langle \mathcal{F}^d \rangle$. We write ϕ for the canonical generator of this finite group and form the semi-direct

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products $\tilde{\mathrm{G}}(k_d)_i := \mathrm{G}(k_d) \rtimes \langle\phi^i\rangle$ and $\tilde{\mathrm{G}}(k_d) := \tilde{\mathrm{G}}(k_d)_1$. For $g \in \mathrm{G}(k_d)$ we write ${}^\phi g := \phi g \phi^{-1}$ and for S any $\langle\phi\rangle$-set we write $S^{\langle\phi\rangle} := \{s \in S \,|\, {}^\phi s = s\}$. For Σ a representation of a $\langle\phi\rangle$-stable subgroup $Y \subset \mathrm{G}_m(k_d)$ we define the transform ${}^\phi\Sigma$ by setting ${}^\phi\Sigma({}^\phi g) = \Sigma(g)$ $(g \in Y)$.

§0.2 Shintani's Theorem.

For each integer i the automorphism ϕ^i of $\mathrm{G}(k_d)$ gives rise to a matrix norm on $\mathrm{G}(k_d)$:

$$g \mapsto \mathcal{N}_i(g) := (g\phi^i)^{d/(d,i)} \in \mathrm{G}(k_d). \tag{0.1}$$

In effect, the matrix norm mapping $\mathcal{N}_i$ sends conjugacy classes of $\tilde{\mathrm{G}}(k_d)$ which are contained in the coset $\mathrm{G}(k_d)\phi^i$ to conjugacy classes in $\mathrm{G}(k_d)$ which have non-void intersection with $\mathrm{G}(k_{(d,i)})$. We interpret this as a mapping from $\mathrm{G}(k_d)$ to $\mathrm{G}(k_d)$. Since the intersection of a conjugacy class of $\mathrm{G}(k_d)$ with $\mathrm{G}(k_{(d,i)})$ is either empty or a conjugacy class of $\mathrm{G}(k_{(d,i)})$, the matrix norm $\mathcal{N}_i$ induces a mapping from conjugacy classes of $\tilde{\mathrm{G}}(k_d)$ contained in the coset $\mathrm{G}(k_d)\phi^i$ to conjugacy classes of $\mathrm{G}(k_{(d,i)})$. We write $[\mathcal{N}_i(g)]$ for the intersection of the conjugacy class of $\mathcal{N}_i(g)$ with $\mathrm{G}(k_{(d,i)})$.

Now let Σ be an irreducible unitary representation of $\mathrm{G}(k_d)$ and let Θ_Σ denote the character of Σ. We say that Σ or Θ_Σ is $\mathrm{Gal}(k_d|k_{(d,i)})$-*invariant* or $\langle\phi^i\rangle$-*invariant* if $\Theta_\Sigma({}^{\phi^i}g) = \Theta_\Sigma(g)$ for all $g \in \mathrm{G}(k_d)$. In this case, since $\langle\phi^i\rangle$ is a finite cyclic group, we have $d/(d,i)$ inequivalent extensions of Σ to $\tilde{\mathrm{G}}(k_d)_i$.

We assume the following fundamental result of Shintani:

Theorem [SH, Theorem 1]**.** *To each pair (Σ, ϕ^i), where $\phi^i \in \langle\phi\rangle$ $(0 \le i < d)$ and Σ is a $\langle\phi^i\rangle$-invariant irreducible representation of $G(k_d)$, corresponds a unique pair $(\tilde\Sigma_i, j_i(\Sigma))$, where $\tilde\Sigma_i$ is an extension of Σ to $\tilde G(k_d)_i$ and $j_i(\Sigma)$ is a representation of $G(k_{(d,i)})$, such that the character relation*

$$\Theta_{\tilde\Sigma_i}(g\phi^i) = \Theta_{j_i(\Sigma)}([\mathcal{N}_i(g)]) \tag{0.2}$$

is satisfied by all $g \in G(k_d)$. The mapping $\Sigma \mapsto j_i(\Sigma)$ from $\langle\phi^i\rangle$-invariant irreducible characters of $G(k_d)$ to irreducible characters of $G(k_{(d,i)})$ is a bijection.

Remark 0.1. Since $\Theta_{\tilde\Sigma_i}(\phi^i) = \Theta_{j_i(\Sigma)}(I) = \dim(j_i(\Sigma))$, it follows that the extension $\tilde\Sigma_i$ has to be the unique extension such that $\Theta_{\tilde\Sigma_i}(\phi^i) > 0$.

The representation $j_i(\Sigma)$ of $\mathrm{G}(k_{(d,i)})$ is called the *Shintani descent of* Σ corresponding to ϕ^i. Since $\Sigma \mapsto j_i(\Sigma)$ is a bijection, there is an inverse mapping bc_i, which sends the set of irreducible characters of $\mathrm{G}(k_{(d,i)})$ bijectively to the set of $\langle\phi^i\rangle$-invariant irreducible characters of $\mathrm{G}(k_d)$. The mapping $bc_i : \mathrm{G}(k_{(d,i)})^\wedge \to \mathrm{G}(k_d)^\wedge_{\langle\phi^i\rangle-\mathrm{inv}}$ is called the *base change lift* mapping.

At least in the case of cuspidal Σ, these mappings are known to be independent of i: $j_i = j_{i'}$ and $bc_i = bc_{i'}$ for $(d,i) = (d,i')$. Moreover, compositions of descent (base change lift) mappings also are descent (base change lift) mappings. The known proofs of these facts rely on global arguments, i. e. use of the Selberg trace formula. The purpose of this paper is to present proofs, or rather verifications, of these facts for cuspidal representations of finite general linear groups which depend only upon Shintani's Theorem and the very classical unitary representation theory of finite general linear groups. In particular, we use results of J. A. Green and S. I. Gelfand, but do not use Deligne/Lusztig theory.

In §0.3 we motivate our main results, Theorems 1 and 2, by reviewing related aspects of the representation theory of Deligne/Weil groups. In §0.4 we give the statements of Theorems 1 and 2. Section 1 proves that the descent of a cuspidal representation is cuspidal (Theorem 1) and Section 2 determines the Green's parameter of the descent (Theorem 2).

Shintani stated the GL_2 case of Theorems 1 and 2 in his paper [SH], which formulates the notion of descent as a concept in representation theory. In [LA] Langlands used the trace formula to prove general theorems concerning base-change lifts of representations of GL_2 over local fields and constructed non-trivial examples of L-functions which satisfy Artin's conjecture. Arthur and Clozel [AC] obtained far more general results than our finite field assertions for GL_n over local fields, using the trace formula for their proofs.

§0.3 The "Perfect Square".

Let $K|F$ be a finite extension of local fields, let k_d denote the residual field of K and k the residual field of F. Let $W'_K \hookrightarrow W'_F$ denote the morphism of Deligne/Weil groups which corresponds to the extension $K|F$. It is widely expected that we shall have the following commutative diagram

$$\begin{array}{ccc} \{\text{irred. admiss. reps. of } \mathrm{GL}_n(F)\} & \longrightarrow & \{n\text{-dim. reps. of } W'_F\} \\ {\scriptstyle bc}\downarrow & & {\scriptstyle res}\downarrow \\ \{\text{irred. admiss. reps. of } \mathrm{GL}_n(K)\} & \longrightarrow & \{n\text{-dim. reps. of } W'_K\}, \end{array} \tag{0.3}$$

in which the horizontal arrows are bijections, bc denotes the base-change lift mapping, and res denotes the restriction mapping induced by $W'_K \hookrightarrow W'_F$.

Now assume that $F_d|F$ is an unramified extension of degree d. In this situation, the parameterization of Macdonald [MA] suggests that, induced by (0.3), there should be a diagram

$$\begin{array}{ccc} \{\text{irred. unit. reps. of } \mathrm{GL}_n(k)\} & \longrightarrow & \{n\text{-dim. tame reps. of } W'_F\}/\sim \\ {\scriptstyle bc}\downarrow & & {\scriptstyle res}\downarrow \\ \{\text{irred. unit. reps. of } \mathrm{GL}_n(k_d)\} & \longrightarrow & \{n\text{-dim. tame reps. of } W'_{F_d}\}/\sim . \end{array} \tag{0.4}$$

In (0.4) bc denotes the Shintani base change lift mapping from the set of irreducible unitary representations of $\mathrm{GL}_n(k)$ to the set of $\langle\phi\rangle$-invariant irreducible unitary representations of $\mathrm{GL}_n(k_d)$, and res again denotes the restriction mapping induced by $W'_{F_d} \hookrightarrow W'_F$. We indicate by "$\sim$" that we identify on the right side representations of Deligne/Weil groups which differ by unramified twists.

Consider the diagram of fields

$$\begin{array}{ccc} k & \longleftarrow & k_n \\ \uparrow & & \uparrow \\ k_d & \longleftarrow & k_{dn}, \end{array} \tag{0.5}$$

where the arrows represent norm mappings. To any $\langle\mathcal{F}\rangle$-orbit of regular $k_n|k$ characters $[\chi_n] \subset X(k_n^\times)$ Green has associated a cuspidal representation $\Pi_{\chi_n} := \Pi_{[\chi_n]}$ of $\mathrm{GL}_n(k)$. On the dual side, denoting the inertial subgroup of W_F' by I_F, we have the surjections

$$I_F \twoheadrightarrow k_n^\times \twoheadrightarrow k^\times$$

and therefore also the induced irreducible n-dimensional representation

$$\Phi_{\chi_n} := \mathrm{Ind}_{W'_{F_n}}^{W'_F} \hat{\chi}_n,$$

where $\hat{\chi}_n$ is any tame character of W'_{F_n} which is constructed by inflation and extension from $\chi_n \in X(k_n^\times)$. Since any two such characters $\hat{\chi}_n$ differ by an unramified twist and since $\Phi_{[\chi_n]} := \Phi_{\chi_n}$ also depends only upon the Galois orbit of χ_n, we have constructed a natural candidate for the image of Π_{χ_n} under the morphism represented by the top horizontal arrow of (0.4).

If $(d,n) = 1$, we obtain the diagram of fields (0.5) to which we give the name "perfect square". Let $\mathrm{N}^*_{k_{dn}|k_d}(\chi_n) = \chi_{dn} \in X(k_{dn}^\times)$ be the image of the regular character χ_n under the dual of the norm mapping $\mathrm{N}_{k_{dn}|k_n} : k_{dn}^\times \to k_n^\times$. Then χ_{dn} is regular with respect to $k_{dn}|k_d$, so we have both the cuspidal representation $\Pi_{\chi_{dn}}$ of $\mathrm{G}_n(k_d)$ and the family of representations $\Phi_{\chi_{dn}} = \mathrm{Ind}_{W'_{F_{dn}}}^{W'_{F_d}} \hat{\chi}_{dn}$ of W'_{F_d}, which is associated to $\Pi_{\chi_{dn}}$ by the bottom horizontal arrow of (0.4). Since restriction is the canonical morphism of $W'_{F_d} \hookrightarrow W'_F$ and since

$$\Phi_{\chi_{dn}} = \Phi_{\chi_n}|W'_{F_d},$$

the restrictions of unramified twists being unramified twists, (0.4) should imply that

$$bc(\Pi_{\chi_n}) = \Pi_{\chi_{dn}}. \tag{0.6}$$

The purpose of this paper is to verify (0.6). In the next section we shall give a precise formulation of our main results.

§0.4 Statements of the Principal Theorems.

Fix two positive integers d, n. A character $\chi_{dn} \in X(k_{dn}^\times)$ is called *k_d-regular* if the $\mathrm{Gal}(k_{dn}|k_d)$-orbit $[\chi_{dn}]$ of χ_{dn} has length n.

As already noted, the work [GR] of J. A. Green gives a bijective correspondence

$$\mathrm{Gal}(k_{dn}|k_d)\backslash X(k_{dn}^\times)_{k_d\text{-reg}} \longleftrightarrow G_n(k_d)^\wedge_{cusp} \tag{0.7}$$

between the set of $\mathrm{Gal}(k_{dn}|k_d)$-orbits of k_d-regular characters of $k_{dn}^\times$ and the set of irreducible cuspidal representations of $G_n(k_d)$. Green's correspondence commutes with the action of $\mathrm{Gal}(k_d|k)$, which acts on both sides.

Assume that $[\chi_{dn}]$ is a $\mathrm{Gal}(k_{dn}|k_d)$-orbit of k_d-regular characters of $k_{dn}^\times$. Let $\Pi := \Pi_{\chi_{dn}}$ denote the corresponding cuspidal representation of $G_n(k_d)$.

Lemma 0.1. *The following statements are equivalent:*

(i) *The cuspidal representation* Π *is* $Gal(k_d|k)$*-invariant.*

(ii) *The character* χ_{dn} *factors through the norm mapping* $N_{k_{dn}|k_n}$, *i. e.,* $\chi_{dn} = \chi_n \circ N_{k_{dn}|k_n}$ *for some* k*-regular character* $\chi_n \in X(k_n^\times)$, *and* $(d,n) = 1$.

Proof. Since (0.7) is $\mathrm{Gal}(k_d|k)$-equivariant, (ii) implies (i). Conversely, if the cuspidal representation Π is $\mathrm{Gal}(k_d|k)$-invariant, the orbit $[\chi_{dn}]$ is $\mathrm{Gal}(k_d|k)$-stable, which implies that $\chi_{dn}^q = \chi_{dn}^{q^{di}}$ for some positive integer i. Therefore, $\chi_{dn}^{q^j} = \chi_{dn}^{q^{dji}}$ for all integers j and, in particular,

$$\chi_{dn}^{q^n} = \chi_{dn}^{q^{dni}} = \chi_{dn}. \tag{0.8}$$

This proves that the $\mathrm{Gal}(k_d|k)$-orbit of χ_{dn} has order dividing n. If $(d,n) > 1$, then

$$\chi_{dn}^{q^{n/(d,n)}} = \chi_{dn}^{q^{nid/(d,n)}} = \chi_{dn},$$

which implies that the $\mathrm{Gal}(\bar{k}|k)$ orbit of χ_{dn} has order dividing $n/(d,n)$. Thus χ_{dn} cannot be k_d-regular unless $(d,n) = 1$. From equation (0.8) and the k_d-regularity of χ_{dn} it follows that

$$\chi_{dn} = \chi_n \circ \mathrm{N}_{k_{dn}|k_n}, \tag{0.9}$$

for some k-regular character $\chi_n \in X(k_n^\times)$. $\square$

Now assume that $(d,n) = 1$. In this case, we have a natural isomorphism

$$\mathrm{Gal}(k_{dn}|k_n) \xrightarrow{\sim} \mathrm{Gal}(k_d|k)$$

and we may regard $\phi \in \mathrm{Gal}(k_{dn}|k_n)$ as the shift of the k-Frobenius. Consider the diagram

$$k_{dn}^\times \xrightarrow{\sim} \Gamma(k_d) \hookrightarrow G_n(k_d), \tag{0.10}$$

a $\langle\phi\rangle$-equivariant embedding. This diagram restricts to a diagram

$$k_n^\times \xrightarrow{\sim} \Gamma(k) \hookrightarrow G_n(k) \tag{0.11}$$

and $\langle\phi\rangle$ acts trivially on (0.11).

Now return to the cuspidal representation Π of $\mathrm{G}_n(k_d)$. In this case, we show in Lemma 1.3 essentially that $\tilde{\Pi}_i(\phi^i)$ can be represented as a permutation matrix, that $\Theta_{\tilde{\Pi}_i}(\phi^i)$ is the number of fixed points of this matrix. Thus $\tilde{\Pi}_1(\phi^i) = \tilde{\Pi}_1(\phi)^i$ is a permutation matrix too and $\tilde{\Pi}_1$ restricted to $\tilde{\mathrm{G}}_n(k_d)_i$ may also be regarded as an extension of Π.

Remark 0.2. Since $\tilde{\Pi}_i = \tilde{\Pi}_1|_{\tilde{\mathrm{G}}_n(k_d)_i}$ for all i, we set $\tilde{\Pi} := \tilde{\Pi}_i$ for $(d,i) = 1$. Notice that this does not imply that $j_i(\Pi) = j_{i'}(\Pi)$ for $(d,i) = (d,i')$. In this paper we shall prove this for the cuspidal representation Π (cf. Theorem 2 below).

Remark 0.3. If $\omega_i \in X(k_{(d,i)}^\times)$ denotes the central character of the representation $j_i(\Pi) \in \mathrm{G}_n(k_{(d,i)})^\wedge_{\text{cusp}}$, where $j_0(\Pi) = \Pi$, then

$$\omega_i \circ \mathrm{N}_{k_d|k_{(d,i)}} = \omega_0 \tag{0.12}$$

and

$$\omega_1 \circ \mathrm{N}_{k_{(d,i)}|k} = \omega_i \tag{0.13}$$

for all $i = 0, \dots, d-1$.

From here on we assume that $\Pi \in G_n(k_d)^\wedge$ is cuspidal and that Π is $\langle\phi\rangle$-invariant. From Lemma 0.1 it follows that $(d,n) = 1$ and that $\chi_{dn} = \chi_n \circ N_{k_{dn}|k_n}$ for some regular character $\chi_n \in X(k_n^\times)$. Moreover, (0.8) implies that $[\chi_{dn}]$ is $\langle\mathcal{F}\rangle$-stable.

Theorem 1. *The descent $j_i(\Pi) \in G_n(k_{(d,i)})^\wedge$ is cuspidal and $\langle\phi\rangle$-invariant for all i. More precisely, its Green's parameter $[\chi_{(d,i)n,i}]$ satisfies the relation*

$$\chi_{(d,i)n,i} = \chi_{n,i} \circ N_{k_{(d,i)n}|k_n} \in X(k_{(d,i)n}^\times) \tag{0.14}$$

for some regular $\chi_{n,i} \in X(k_n^\times)$.

Remark 0.4. More generally, we shall prove that the Shintani descent of a generic representation is generic (Corollary 1.5).

Set $\chi_{dn,0} = \chi_{dn}$, $\chi_{n,0} = \chi_n$. Because the central character of $j_i(\Pi)$ is

$$\omega_i = \chi_{(d,i)n,i}|_{k_{(d,i)}^\times} = \chi_{n,i}|_{k^\times} \circ \mathrm{N}_{k_{(d,i)}|k}, \tag{0.15}$$

Remark 0.3 implies that $\chi_{n,i}|_{k^\times} = \omega_1$ for all $i = 0, \dots, d-1$.

Our aim is to prove that $[\chi_{n,i}] = [\chi_{n,0}]$ for all $i = 1, \dots, d-1$, in other words that the cuspidal representation $j_i(\Pi)$ corresponds to $[\chi_n \circ \mathrm{N}_{k_{(d,i)n}|k_n}]$. By Green's theory [GR] this implies that $j_i(\Pi) = j_{i'}(\Pi)$ if $(d,i) = (d,i')$.

Here is the precise statement of our result:

Theorem 2. *Let $\Pi := \Pi_{\chi_{dn}} \in G_n(k_d)^\wedge_{\text{cusp}}$ correspond to the orbit of k_d-regular characters $[\chi_{dn}] \subset X(k_{dn}^\times)$. Assume that Π is $\langle\phi\rangle$-invariant, i. e., that there is a regular orbit*

$$[\chi_n] = Gal(k_n|k) \cdot \chi_n \subset X(k_n^\times)$$

such that $\chi_{dn} = \chi_n \circ N_{k_{dn}|k_n}$. Let $j_i(\Pi) \in G_n(k_{(d,i)})^\wedge_{\text{cusp}}$ be the Shintani descent of Π corresponding to ϕ^i. Let $[\chi_{(d,i)n,i}] \subset X(k_{(d,i)n}^\times)$ be the Green's parameter of $j_i(\Pi)$. Then

$$[\chi_n \circ N_{k_{(d,i)n}|k_n}] = [\chi_{(d,i)n,i}]. \tag{0.16}$$

The following assertions are immediate consequences of Theorem 2:

Corollary 3. *Let Π be a cuspidal representation of $G_n(k_d)$.*

(1) *The Shintani descent $j_{k_d|k_{(d,i)}}(\Pi)$ is independent of the choice of generator of the Galois group $Gal(k_d|k_{(d,i)})$ in the sense that*

$$j_{k_d|k_{(d,i)}}(\Pi) := j_i(\Pi) = j_{i'}(\Pi). \tag{0.17}$$

for $(d, i') = (d, i)$.

(2) *Let $k_d|k_{(d,i)}|k$ be any tower of fields. Then*

$$j_{k_d|k}(\Pi) = j_{k_{(d,i)}|k}(j_{k_d|k_{(d,i)}}(\Pi)). \tag{0.18}$$

Remark 0.5. [KA, (1.3.5)(iii)] points out that, if $(d, |\mathrm{G}(k_d)|) = 1$, then a version of (0.18) follows from work of Glauberman [GL].

The proofs of Theorems 1 and 2 will be presented in the two concluding sections of the paper.

1. The Descent, j_i, of a $\langle\phi\rangle$-invariant Cuspidal Representation is $\langle\phi\rangle$-invariant and Cuspidal

Our goal in this section is to prove Theorem 1, first, by showing that the descent of a cuspidal representation is cuspidal, then by proving Lemma 1.6, which justifies (0.14). In proving that the descent of a cuspidal representation is cuspidal it is enough to treat the case $i = 1$. We write $j(\Pi) := j_1(\Pi)$.

Let U_0 denote the upper triangular unipotent subgroup, T_0 the diagonal subgroup, $Z \subset T_0$ the scalar subgroup, and $B := T_0 \ltimes U_0$ the upper triangular subgroup of $G = G_n$ as algebraic groups.

Consider the maximal parabolic subgroup $\mathcal{P} := (\mathrm{G}_{n-1} \times \mathrm{G}_1) \ltimes U$ of G which contains B and has the block diagonal group $G_{n-1} \times G_1$ as a Levi subgroup. The unipotent radical U of $\mathcal{P}$ is the subgroup of U_0 which has non-zero off-diagonal entries only in the last column. For any $\ell \geq 1$ the group $U(k_\ell)$ is isomorphic to the additive group $(k_\ell^{n-1})^+$. We shall work with the Gelfand group $\mathcal{Q} := (\mathrm{G}_{n-1} \times 1) \ltimes U$, where $\mathcal{P} = \mathcal{Q} \times Z$. It is easy to pass between $\mathcal{P}$ and $\mathcal{Q}$ and statements of results which we shall apply are simpler when they are formulated in terms of $\mathcal{Q}$.

Let $\ell \geq 1$. A character $\psi \in X(U_0(k_\ell))$ is called *generic* if its restriction $I + x_{i,i+1} \mapsto \psi(I + x_{i,i+1})$ to every one-parameter superdiagonal subgroup $(1 \leq i < n)$ is non-trivial. The set of generic characters of $U_0(k_\ell)$ comprise a single orbit under the coadjoint action of $T_0(k_\ell)$ on $X(U_0(k_\ell))$. A representation Π of $\mathrm{G}(k_\ell)$ is called *generic* if its restriction $\Pi|_{U_0(k_\ell)}$ contains a generic character.

The following result, essentially due to S. I. Gelfand, provides a key tool which we shall apply throughout this section.

Lemma 1.1 [G1,G2]**.** *Let $\ell \geq 1$. Let ψ be any generic character of $U_0(k_\ell)$. Let*

$$\rho := \rho_n := \rho_n(k_\ell) := Ind_{U_0(k_\ell)}^{\mathcal{Q}(k_\ell)}(\psi). \tag{1.1}$$

Then ρ is an irreducible $\prod_{j=1}^{n-1}(q^{j\ell} - 1)$-dimensional representation of $\mathcal{Q}(k_\ell)$ which is independent of the choice of generic character ψ. Let Σ be an irreducible representation of $G(k_\ell)$. Then Σ is generic if and only if $\Sigma|_{\mathcal{Q}(k_\ell)}$ contains ρ and Σ is *cuspidal if and only if* $\Sigma|_{\mathcal{Q}(k_\ell)} \sim \rho$. *In particular, every cuspidal representation of* $G(k_\ell)$ *is generic.*

Proof. It is enough to consider the case $\ell = 1$. Since

$$[\mathcal{Q}(k) : U_0(k)] = \prod_{j=1}^{n-1}(q^j - 1), \tag{1.2}$$

this is the degree of ρ. Moreover,

$$[B(k) \cap \mathcal{Q}(k) : U_0(k)] = (q-1)^{n-1}, \tag{1.3}$$

which is also the number of generic characters of $U_0(k)$. Since $T_0(k)$, acting via conjugation, acts transitively on the set of generic characters of $U_0(k)$, it follows that the representation $\mathrm{Ind}_{U_0(k)}^{B(k)\cap\mathcal{Q}(k)}\psi$ is irreducible and independent of the choice of ψ. From the transitivity of induction it follows that ρ does not depend upon ψ. We refer to either [G1, G2] or [SZ, Lemma 5.3], for a proof of the irreducibility of ρ. Every cuspidal representation is generic ([SZ, Lemma 5.2], for instance); moreover, if Σ is an irreducible generic representation of $\mathrm{G}(k)$, then, by Frobenius, $\Sigma|_{\mathcal{Q}(k)}$ contains ρ. Green ([GR, p. 431, the third equation]) has proved that the dimension of any irreducible cuspidal representation of $\mathrm{G}(k)$ is $\prod_{j=1}^{n-1}(q^j-1)$, which is the same as the dimension of ρ, so if Σ is irreducible and cuspidal, then $\Sigma|_{\mathcal{Q}(k)} \sim \rho$. The converse is obvious. $\square$

Now let ψ_1 be a non-trivial additive character of k. Then the character $\psi_d := \psi_1 \circ \mathrm{tr}_{k_d|k}$ is a non-trivial additive character of k_d which is $\langle\phi\rangle$-*invariant* in the sense that $\psi_d \circ \phi = \psi_d$. Define the $\langle\phi\rangle$-invariant generic character $\psi \in X(U_0(k_d))$ by setting $\psi(I+x) = \psi_d(x_{1,2} + \cdots + x_{n-1,n})$ for any $I + x \in U_0(k_d)$. Extend ψ to a character $\tilde{\psi}$ of $\tilde{U}_0(k_d) := \langle\phi\rangle \ltimes U_0(k_d)$ by setting $\tilde{\psi}(\phi) = 1$.

Lemma 1.2. *Let* $\tilde{\psi}$ *be the character of* $\tilde{U}_0(k_d)$ *which is defined above. Let*

$$\tilde{\rho} := \tilde{\rho}_n := Ind_{\tilde{U}_0(k_d)}^{\tilde{\mathcal{Q}}(k_d)}(\tilde{\psi}). \tag{1.4}$$

Then $\tilde{\rho}$ *is an irreducible* $\prod_{j=1}^{n-1}(q^{jd}-1)$*-dimensional representation of* $\tilde{\mathcal{Q}}(k_d) := \langle\phi\rangle \ltimes \mathcal{Q}(k_d)$ *which does not depend upon the choice of the* $\langle\phi\rangle$*-invariant generic character* ψ. *Let* Σ *be a* $\langle\phi\rangle$*-invariant irreducible representation of* $G(k_d)$. *Then* Σ *is generic if and only if there is an extension* $\widehat{\Sigma}$ *of* Σ *to* $\tilde{G}(k_d)$ *such that* $\widehat{\Sigma}|_{\tilde{\mathcal{Q}}(k_d)}$ *contains* $\tilde{\rho}$ *and* Σ *is cuspidal if and only if there is an extension* $\widehat{\Sigma}$ *of* Σ *to* $\tilde{G}(k_d)$ *such that* $\widehat{\Sigma}|_{\tilde{\mathcal{Q}}(k_d)} \sim \tilde{\rho}$.

Proof. First it is obvious as before that the dimension of $\tilde{\rho}$ is $\prod_{j=1}^{n-1}(q^{jd}-1)$, as this is again the index $[\tilde{\mathcal{Q}}(k_d) : \tilde{U}_0(k_d)]$. Clearly, the restriction $\tilde{\rho}|_{\mathcal{Q}(k_d)} \sim \rho$; thus $\tilde{\rho}$ is irreducible, since ρ is irreducible. That $\tilde{\rho}$ is independent of the choice of ψ follows immediately from Lemma 1.1, since $\tilde{\rho}|_{\mathcal{Q}(k_d)} = \rho$. If Σ is generic, then, again by Lemma 1.1, $\Sigma|_{\mathcal{Q}(k_d)}$ contains ρ and therefore, assuming that Σ is also

$\langle\phi\rangle$-invariant, any extension of Σ to $\tilde{\mathrm{G}}(k_d)$ contains an extension of ρ. Since for Σ a generic representation of $\mathrm{G}(k_d)$ the restriction $\Sigma|_{\mathcal{Q}(k_d)}$ contains ρ with multiplicity one (cf. [SZ, Corollary 5.6]), $\Sigma|_{U_0(k_d)}$ contains every $\langle\phi\rangle$-invariant generic character ψ of $U_0(k_d)$ also with multiplicity one. Let $\widehat{\Sigma}'$ be an extension of Σ to $\tilde{\mathrm{G}}(k_d)$ and let $0 \neq v_\psi$ be a vector in the representation space V of Σ which transforms under $\Sigma|_{U_0(k_d)}$ as ψ. Then v_ψ is an eigenvector for $\widehat{\Sigma}'(\phi)$ with eigenvalue a d-th root of one. Tensoring $\widehat{\Sigma}'$ with the inflation to $\tilde{\mathrm{G}}(k_d)$ of a suitably chosen character of $\langle\phi\rangle$, we obtain an extension $\widehat{\Sigma}$ with the required properties. The remaining statements of the Lemma follow immediately from Lemma 1.1. □

For any generic representation Σ of $\mathrm{G}(k_d)$, let $\widehat{\Sigma}$ denote the extension of Σ to $\tilde{\mathrm{G}}(k_d)$ such that $\widehat{\Sigma}|_{\tilde{U}_0(k_d)}$ contains $\tilde{\rho}$. We also write $\Theta_{\tilde{\rho}}$ for the character of $\tilde{\rho}$. The gist of Lemma 1.3 is that $\widehat{\Pi} = \tilde{\Pi}$, the extension of Π which occurs in equation (0.2); for all i the descents of Π correspond to this one extension (see Corollary 1.5 for the more general statement which applies to all generic representations).

Lemma 1.3. *Let Π be a $\langle\phi\rangle$-invariant cuspidal representation of $G(k_d)$. Then $\Theta_{\widehat{\Pi}}(g\phi) = \Theta_{j(\Pi)}([\mathcal{N}(g)])$ for all $g \in G(k_d)$, so $\widehat{\Pi} = \tilde{\Pi}$. In particular, for all $p \in \mathcal{Q}(k_d)$, $\Theta_{\tilde{\rho}}(p\phi) = \Theta_{j(\Pi)}([\mathcal{N}(p)])$, $\Theta_{\tilde{\rho}}(\phi) = dim(j(\Pi)) = \prod_{j=1}^{n-1}(q^j-1)$.*

Proof. Remark 0.1 implies that it is enough to show that $\Theta_{\widehat{\Pi}}(\phi) > 0$. Since $\widehat{\Pi}|_{\tilde{\mathcal{Q}}} \sim \tilde{\rho}$, we have $\Theta_{\widehat{\Pi}}(p\phi) = \Theta_{\tilde{\rho}}(p\phi)$ for all $p \in \mathcal{Q}(k_d)$. By Lemma 1.2 and Frobenius' formula, when $p = I$,

$$\Theta_{\tilde{\rho}}(\phi) = \sum_{x \in \tilde{\mathcal{Q}}(k_d)/\tilde{U}_0(k_d)} \tilde{\psi}(x^{-1}\phi x) = \sum_{x \in \mathcal{Q}(k_d)/U_0(k_d)} \tilde{\psi}(x^{-1}\,{}^{\phi}x\phi), \tag{1.5}$$

where $\tilde{\psi}$ is extended to $\tilde{\mathcal{Q}}(k_d)$ by zero on $\tilde{\mathcal{Q}}(k_d) - \tilde{U}_0(k_d)$. Clearly, for any $x \in \mathcal{Q}(k_d)$, we have $x^{-1}\,{}^{\phi}x \in U_0(k_d)$ if and only if $x \in \mathcal{Q}(k)U_0(k_d)$, in which case $\tilde{\psi}(x^{-1}\,{}^{\phi}x\phi) = \psi(x^{-1}\,{}^{\phi}x) = 1$. Thus,

$$\Theta_{\widehat{\Pi}}(\phi) = [\mathcal{Q}(k) : U_0(k)] = \dim(\rho_n(k)) = \dim(j(\Pi)), \tag{1.6}$$

which is the positive integer $\prod_{j=1}^{n-1}(q^j-1)$. Since (0.2) uniquely determines $\tilde{\Pi}$, it is clear that $\widehat{\Pi} = \tilde{\Pi}$. □

It follows from Lemma 1.1 that, to prove that $j(\Pi)$ is cuspidal, it suffices to prove that $j(\Pi)$ is generic. As Proposition 1.4 clearly implies that $j(\Pi)$ is generic when Π is cuspidal, the proof of Proposition 1.4 will complete the proof that $j(\Pi)$ is cuspidal.

Proposition 1.4. *For all $p \in \mathcal{Q}(k_d)$*

$$\Theta_{\tilde{\rho}_n}(p\phi) = \Theta_{\rho_n(k)}([\mathcal{N}(p)]). \tag{1.7}$$

We shall devote §1.1 to the proof of Proposition 1.4. Although Proposition 1.4 and Lemma 1.1 imply that the descent of a cuspidal representation is cuspidal, the proof that the descent of an arbitrary generic representation is generic needs a subtler argument.

Corollary 1.5. *Let Σ be generic and $\langle\phi\rangle$-invariant. Then $j(\Sigma)$ is also generic. Moreover, for all $g \in G(k_d)$*

$$\Theta_{\widehat{\Sigma}}(g\phi) = \Theta_{j(\Sigma)}([\mathcal{N}(g)]),$$

$\widehat{\Sigma}$ being, as defined above, the unique irreducible extension of Σ to $\tilde{G}(k_d)$ which contains $\tilde{\rho}_n$ in its restriction to $\tilde{\mathcal{Q}}_n(k_d)$.

Proof. According to [SH, Corollary to Lemma 2-6(ii)] the mapping j from the space of $\tilde{\mathcal{Q}}(k_d)$-class functions on the coset $\mathcal{Q}(k_d)\phi$ to the space of class functions on $\mathcal{Q}(k)$ is an isometry. (Shintani stated this for parabolic subgroups, but by Lemma 1.1.2 and Remark 1.1 the isometry for the parabolic group $Z \times \mathcal{Q}$ induces a similar isometry for $\mathcal{Q}$.) Using Proposition 1.4 and this isometry property, we obtain

$$\begin{aligned}\langle\Theta_{j(\Sigma)}, \Theta_{\rho_n(k)}\rangle_{\mathcal{Q}(k)} &= \langle\Theta_{\tilde{\Sigma}}, \Theta_{\tilde{\rho}_n}\rangle_{\mathcal{Q}(k_d)\phi} \\ &= \frac{1}{|\mathcal{Q}(k_d)|}\sum_{p\in\mathcal{Q}(k_d)} \Theta_{\tilde{\Sigma}}(p\phi)\bar{\Theta}_{\tilde{\rho}_n}(p\phi) \\ &= \sum_{\xi\in X(\langle\phi\rangle)} \langle\tilde{\Sigma}, \tilde{\rho}_n\xi\rangle_{\tilde{\mathcal{Q}}(k_d)} \cdot \xi(\phi).\end{aligned}$$

The verification of the last equality is left to the reader. For this last equality we are regarding class functions on $\mathcal{Q}(k_d)\phi$ as restrictions of class functions on $\tilde{\mathcal{Q}}(k_d)$. In a more general context such an equality has been used by Digne/Michel [DM].

Now assume that Σ is generic. Then we know that $\Sigma|_{\mathcal{Q}(k_d)}$ contains ρ_n with multiplicity one and therefore $\tilde{\Sigma}|_{\tilde{\mathcal{Q}}(k_d)}$ contains one and only one extension $\tilde{\rho}_n\xi_1$ of ρ_n to $\tilde{\mathcal{Q}}(k_d)$ for some character $\xi_1 \in X(\langle\phi\rangle)$ and this extension also occurs simply. Hence the right side of our sequence of equalities is $\xi_1(\phi)$. Since the left side is a non-negative integer, it follows that $\xi_1 \equiv 1$ and $\langle\Theta_{j(\Sigma)}, \Theta_{\rho_n(k)}\rangle = 1$. Therefore, $\tilde{\Sigma} = \hat{\Sigma}$ and $j(\Sigma)$ is generic. $\square$

To complete the proof of Theorem 1 we have to prove that $j_i(\Pi)$ is $\langle\phi\rangle$-invariant:

Lemma 1.6. *Let Σ be an irreducible and $\langle\phi^i\rangle$-invariant representation of $G(k_d)$. Then $j_i({}^{\phi}\Sigma) ={}^{\phi} j_i(\Sigma)$ for all integers i. In particular, if Σ is $\langle\phi\rangle$-invariant, then so is $j_i(\Sigma)$.*

Proof. It is enough to consider the case in which $i = 1$, so we omit the index i. By (0.2),

$$\Theta_{j({}^{\phi}\Sigma)}([\mathcal{N}(g)]) = \Theta_{\widetilde{{}^{\phi}\Sigma}}(g\phi)$$

and, by (0.1) and (0.2),

$$\Theta_{{}^{\phi}j(\Sigma)}([\mathcal{N}(g)]) = \Theta_{j(\Sigma)}({}^{\phi^{-1}}[\mathcal{N}(g)]) = \Theta_{j(\Sigma)}([\mathcal{N}({}^{\phi^{-1}}g)]) = \Theta_{\tilde{\Sigma}}({}^{\phi^{-1}}g\phi).$$

Since

$$\widetilde{{}^{\phi}\Sigma}(g\phi) = \tilde{\Sigma}(\phi^{-1}g\phi^2) = {}^{\phi}\tilde{\Sigma}(g\phi),$$

it follows that $j({}^{\phi}\Sigma) \sim {}^{\phi}j(\Sigma)$. Furthermore, if ${}^{\phi}\Sigma \sim \Sigma$, then $j(\Sigma) \sim {}^{\phi}j(\Sigma)$. $\square$

§1.1 The Proof of Proposition 1.4.

We consider a non-ordered partition $\lambda = \lambda_1, \ldots, \lambda_p$ of n consisting of $p = p(\lambda)$ parts and the corresponding block-diagonal matrix

$$u_\lambda(1) := \mathrm{diag}(u_{(\lambda_1)}(1), \ldots, u_{(\lambda_p)}(1)),$$

where $u_{(\lambda_i)}$ denotes the standard upper-triangular $\lambda_i \times \lambda_i$ unipotent Jordan companion matrix. We also define $u_\lambda(x)$ to be a unipotent matrix with the same support as $u_\lambda(1)$ but with the more general super-diagonal vector $x = (x_{12}, \ldots, x_{n-1,n})$, where $x_{i,i+1} \in k_d$. If, in addition, $x_{i,i+1} \neq 0$ if and only if $\mathrm{tr}_{k_d|k}(x_{i,i+1}) \neq 0$ for all i, $1 \leq i < n$, we call $u_\lambda(x)$ a *special Jordan matrix.*

We reformulate Proposition 1.4 as:

Proposition 1.1.1.

(i) *For all $p \in \mathcal{Q}(k_d)$*

$$\Theta_{\tilde{\rho}_n}(p\phi) = \Theta_{\rho_n(k)}([\mathcal{N}(p)]),$$

where $[\mathcal{N}(p)]$ denotes the conjugacy class of $\mathcal{N}(p) = (p\phi)^d$ intersected with $\mathcal{Q}(k_d)$.

(ii) *For all special Jordan matrices $u_\lambda(x) \in U_0(k_d)$*

$$\Theta_{\tilde{\rho}_n}(u_\lambda(x)\phi) = \Theta_{\rho_n(k)}(u_\lambda(y)),$$

where $y_{i,i+1} = tr_{k_d|k}(x_{i,i+1})$.

Proof. (ii) is a priori a special case of (i), since $u_\lambda(y)$ is a representative for the conjugacy class $[\mathcal{N}(u_\lambda(x))]$ (cf. Corollary 1.1.9), assuming the above introduced notation for $u_\lambda(x)$ and $u_\lambda(y)$. □

Now we proceed to prove Proposition 1.1.1 via an induction on n, arguing that

$$(i)_{n-1} \Rightarrow (ii)_n \Rightarrow (i)_n.$$

First, let us prove the implication $(ii)_n \Rightarrow (i)_n$. We have to modify Shintani's Lemma 2-6, which is formulated for parabolic subgroups of G_n, to check that the same Lemma is valid for Gelfand's group $\mathcal{Q}$.

Lemma 1.1.2.

(i) *Shintani's bijection of conjugacy classes*

$$\mathcal{P}(k_d)\backslash\mathcal{P}(k_d)\phi \to \mathcal{P}(k)\backslash\mathcal{P}(k), \qquad p\phi \mapsto [(p\phi)^d] \tag{1.1.1}$$

induces a bijection of conjugacy classes

$$\mathcal{Q}(k_d)\backslash\mathcal{Q}(k_d)\phi \to \mathcal{Q}(k)\backslash\mathcal{Q}(k). \tag{1.1.2}$$

(ii) *Moreover, concerning the left side of this bijection we have*

$$\mathcal{Q}(k_d)\backslash\mathcal{Q}(k_d)\phi = \tilde{\mathcal{Q}}(k_d)\backslash\mathcal{Q}(k_d)\phi \tag{1.1.3}$$

Proof of (i). Since $\mathcal{P}(k_d) = Z(k_d) \times \mathcal{Q}(k_d)$, a direct product, we have $p_i = z_i q_i$ for any $p_1, p_2 \in \mathcal{P}(k_d)$ and therefore

$$\begin{aligned} p_1(p_2\phi)p_1^{-1} &= z_1 q_1 (z_2 q_2 \phi) q_1^{-1} z_1^{-1} \\ &= z_1 z_2 \, {}^{\phi}z_1^{-1} \cdot q_1 (q_2\phi) q_1^{-1}, \end{aligned}$$

since $Z(k_d)$ is central in $\mathcal{P}(k_d)$. Therefore, the direct product $\mathcal{P}(k_d) = Z(k_d) \times \mathcal{Q}(k_d)$ induces a direct product

$$\mathcal{P}(k_d)\backslash\mathcal{P}(k_d)\phi = \{Z(k_d)\underset{\phi}{\backslash}\mathbb{Z}(k_d)\} \times \{\mathcal{Q}(k_d)\backslash\mathcal{Q}(k_d)\phi\}, \tag{1.1.4}$$

where we write $\underset{\phi}{\backslash}$ to indicate the "ϕ conjugation" $z_2 \mapsto z_1 z_2 \, {}^{\phi}z_1^{-1}$. Obviously, for the corresponding sets of conjugacy classes in the respective groups of k-points we have the direct product decomposition

$$\mathcal{P}(k)\backslash\mathcal{P}(k) = Z(k) \times (\mathcal{Q}(k)\backslash\mathcal{Q}(k)) \tag{1.1.5}$$

and the map $p\phi \mapsto [(p\phi)^d]$ preserves the factors in (1.1.4) and (1.1.5), since

$$[(zq\phi)^d] = [(zq)\, {}^{\phi}(zq) \cdots {}^{\phi^{d-1}}(zq)] = (z\phi)^d [(q\phi)^d],$$

inasmuch as $(z\phi)^d \in Z(k)$. Thus the bijection (1) induces bijections between each of the components in (1.1.4) and (1.1.5).

Proof of (ii). Since

$$(p\phi)^d = p \cdot {}^{\phi}p \cdot {}^{\phi^2}p \cdots {}^{\phi^{d-1}}p$$

and

$$({}^{\phi}p\phi)^d = {}^{\phi}p \cdot {}^{\phi^2}p \cdot {}^{\phi^3}p \cdots {}^{\phi^{d-1}}p \cdot p = p^{-1}(p\phi)^d p.$$

Therefore, $[(p\phi)^d] = [({}^{\phi}p\phi)^d]$, so the bijectivity of (1) implies that

$$p\phi \sim {}^{\phi}p\phi = \phi(p\phi)\phi^{-1} \in \mathcal{P}(k_d)\backslash\mathcal{P}(k_d)\phi.$$

Obviously the same argument applies when $\mathcal{P}(k_d)$ is replaced by $\mathcal{Q}(k_d)$. □

Remark 1.1. Using Lemma 1.1.2 let us show that the isometry assertion of [SH, Corollary to Lemma 2-6(ii)] applies to $\mathcal{Q}$ as well as $\mathcal{P}$. For this we define mappings $j_{\mathcal{P}}$ and $j_{\mathcal{Q}}$ from the respective spaces of class functions $\tilde{\mathcal{P}}(k_d)\backslash\mathcal{P}(k_d\phi)$ and $\tilde{\mathcal{Q}}(k_d)\backslash\mathcal{Q}(k_d\phi)$ to the spaces of class functions $\mathcal{P}(k)\backslash\mathcal{P}(k)$ and $\mathcal{Q}(k)\backslash\mathcal{Q}(k)$ such that

$$\Theta(p\phi) = j(\Theta)([(p\phi)^d]).$$

Now, since the map $p\phi \mapsto [(p\phi)^d]$ preserves the factors in (1.1.4) and (1.1.5), we obtain the equality

$$j_{\mathcal{P}}(1 \times \Theta) = 1 \times j_{\mathcal{Q}}(\Theta)$$

for any class function Θ on $\tilde{\mathcal{Q}}(k_d)$. Using Shintani's isometry relation for $\mathcal{P}$ ([SH, Corollary to Lemma 2-6(ii)]), we obtain

$$\begin{aligned}\langle j_{\mathcal{Q}}(\Theta), j_{\mathcal{Q}}(\Psi)\rangle_{\mathcal{Q}(k)} &= \langle 1 \times j_{\mathcal{Q}}(\Theta), 1 \times j_{\mathcal{Q}}(\Psi)\rangle_{\mathcal{P}(k)} \\ &= \langle j_{\mathcal{P}}(1 \times \Theta), j_{\mathcal{P}}(1 \times \Psi)\rangle_{\mathcal{P}(k)} \\ &= \langle 1 \times \Theta, 1 \times \Psi\rangle_{\mathcal{P}(k_d)\phi} \\ &= \langle \Theta, \Psi\rangle_{\mathcal{Q}(k_d)\phi}.\end{aligned}$$

From now on in this section we want to work with the bijection of conjugacy classes

$$\tilde{\mathcal{Q}}(k_d)\backslash\mathcal{Q}(k_d)\phi \to \mathcal{Q}(k)\backslash\mathcal{Q}(k) \tag{1.1.6}$$

defined via the mapping $p\phi \mapsto [(p\phi)^d]$. In Proposition 1.1.1(i), the arguments $p\phi$ and $[\mathcal{N}(p)]$ correspond under this bijection.

Assume that $p\phi$ lies in the support of $\Theta_{\tilde{\rho}_n}$. Since, by Lemma 1.2, $\tilde{\rho}_n$ is induced from $\tilde{\psi}$, it follows that the $\tilde{\mathcal{Q}}(k_d)$-conjugacy class $C := C(p\phi)$ intersects the set $U_0(k_d)\phi$. Similarly, by Lemma 1.1, if $p \in \mathcal{Q}(k)$ lies in the support of $\Theta_{\rho_n(k)}$, then the conjugacy class of p intersects $U_0(k)$.

Lemma 1.1.3.

(i) *The bijection (1.1.6) restricts to a bijection between the set of conjugacy classes which intersect $U_0(k_d)\phi$ and the set of conjugacy classes intersecting $U_0(k)$.*

(ii) *Each $\mathcal{Q}(k)$-conjugacy class consisting of unipotent matrices contains a Jordan matrix $u_\lambda(y) \in U_0(k)$.*

Proof. We refer to Proposition 1.1.4 for the proof of (ii) and prove here the implication $(ii) \Rightarrow (i)$. If C intersects $U_0(k_d)\phi$, then the set of d-th powers C^d is a conjugacy class of unipotent elements in $\mathcal{Q}(k_d)$. It follows from (1.1.6) that C^d contains a single conjugacy class of $\mathcal{Q}(k)$ consisting of unipotent elements. Conversely, if $C^d \cap \mathcal{Q}(k_d)$ intersects $U_0(k)$, then (ii) implies that it contains a Jordan matrix $u_\lambda(y) \in U_0(k)$. From Corollary 1.1.9 and from the injectivity of the bijection (1.1.6) it then follows that there is a special Jordan matrix $u_\lambda(x) \in U_0(k_d)$ such that C contains $u_\lambda(x)\phi$. $\square$

If $C(p\phi)$ does not intersect $U_0(k_d)\phi$, then Lemma 1.1.3 and the bijectivity of the mapping $j_{\mathcal{Q}}$ imply that $C(p\phi)^d$ cannot intersect $U_0(k)$, which implies that both sides of the equation in Proposition 1.1.1(i) vanish. The implication $(ii)_n \Rightarrow (i)_n$ follows from the fact that the set of unipotent conjugacy classes in $\mathcal{Q}(k_d)\phi$ and $\mathcal{Q}(k)$ correspond via the matrix norm map and contain, respectively, elements of the form $u_\lambda(x)\phi$ and $u_\lambda(y)$.

We are left to prove Lemma 1.1.3(ii) and the implication $(ii)_{n-1} \Rightarrow (i)_n$ of Proposition 1.1.1. The following Proposition 1.1.4 implies Lemma 1.1.3(ii).

Proposition 1.1.4. *Every unipotent element of $\mathcal{Q}(k)$ is conjugate in $\mathcal{Q}(k)$ to a Jordan matrix $u_\lambda(y)$ for some partition λ of n. In particular, if $u \in U_0(k)$ has all* *non-zero super-diagonal entries, then* u *is even* $U_0(k)$*-conjugate to* $u_{(n)}$*, where* $u_{(n)}$ *has the same super-diagonal entries as* u*.*

Proof. For the proof of Proposition 1.1.4 it is convenient to represent the "affine group" $\mathcal{Q}_n(k) = k_+^{n-1} \rtimes \mathrm{G}_{n-1}(k)$ by pairs (v, g), where $v = [v_1, \dots, v_{n-1}]^t \in k_+^{n-1}$ and $g \in \mathrm{G}_{n-1}(k)$. In this set-up the multiplication law of $\mathcal{Q}_n(k)$ becomes

$$(w, h) \cdot (v, g) = (w + hv, hg) \qquad (v, w \in k_+^{n-1}; g, h \in \mathrm{G}_{n-1}(k)).$$

Moreover, conjugation takes the form

$$(w, h)(v, g)(w, h)^{-1} = (w + hv, hg)(-h^{-1}w, h^{-1}) = (w + hv - hgh^{-1}w, hgh^{-1}).$$

First let us observe that every unipotent element of $\mathcal{Q}(k)$ is conjugate in $\mathcal{Q}(k)$ to an element of $U_0(k)$. Clearly, if $(v, g) \in \mathcal{Q}(k)$ is unipotent, then g is unipotent. Therefore,

$$(0, h)(v, g)(0, h)^{-1} = (hv, hgh^{-1}),$$

so h can be chosen such that hgh^{-1}, like v, is upper triangular unipotent.

Let $u(x) := I + \sum_{1 \le i < j \le n} x_{ij} e_{ij}$, where $x_{ij} \in k$ and the elements $e_{ij} \in \mathrm{M}_n(k)$ are the canonical basis elements:

$$e_{ij} e_{kl} = e_{il} \delta_{jk}. \tag{1.1.7}$$

Lemma 1.1.5. *If* $u = u(x) \in U_0(k)$*, then* u *is* $\mathcal{Q}(k)$*-conjugate to* $(v, u_{\lambda'}(y))$*, in which* $u_{\lambda'}(y)$ *is a Jordan companion matrix having the super-diagonal vector* $[y_{1,2}, \dots, y_{n-2,n-1}]$*,* $v_{n-1} = x_{n-1,n}$*, and* $y_{i,i+1} v_i = 0$ *for all* i *such that* $1 \le i \le n-2$*. If* $x_{i,i+1} \neq 0$ *for all* $i < n-1$*, then the last statement of Proposition 1.1.4 holds.*

Proof. Using our realization of $\mathcal{Q}(k)$ as an affine group, we write $u = (v, g)$, which belongs to $U_0(k)$ if and only if $g \in U_0'(k) := U_0(k) \cap \mathrm{G}_{n-1}(k)$. By the induction hypothesis we may assume that there exists $h \in \mathcal{Q}_{n-1}(k)$, or even $h \in U_0'(k)$, such that $hgh^{-1} = u_{\lambda'}$ with the super-diagonal vector $y = [y_{12}, \dots, y_{n-2,n-1}]$ or $y = [x_{12}, \dots, x_{n-2,n-1}]$, respectively. Then we obtain

$$(0, h)(v, g)(0, h)^{-1} = (hv, u_{\lambda'}(y)) \quad \text{and} \quad (hv)_{n-1} = v_{n-1} = x_{n-1,n}.$$

Next, conjugating with a unipotent matrix (w, I), we arrive at

$$(w, I)(hv, u_{\lambda'}(y))(w, I)^{-1} = ((I - u_{\lambda'}(y))w + hv, u_{\lambda'}(y)).$$

Since w is arbitrary, $(I - u_{\lambda'}(y))w$ may be chosen as any element of the column space of $I - u_{\lambda'}(y)$, which is a subspace of k^{n-2}. Therefore, $(I - u_{\lambda'}(y))w + hv$ has the same last component $v_{n-1} = x_{n-1,n}$ as previously, and the other coordinates of hv can be changed as asserted. $\square$

Henceforth, we assume that $u = (v, u_{\lambda'}(y))$ and that for v and y we have $y_{i,i+1} v_i = 0$ for all $i \le n-2$.

Lemma 1.1.6. *If there exists $i < n-1$ such that $y_{i,i+1} = 0$ and $v_j = 0$ for all $j \leq i$, then Proposition 1.1.4 is true.*

Proof. Clear. Under these hypotheses we can apply induction. □

Now let $j_0 < n-1$ be such that $v_j = 0$ for all $j < j_0$ and $v_{j_0} \neq 0$. We may also assume that $y_{j_0,j_0+1} = 0$ and $y_{j,j+1} \neq 0$ for $1 \leq j < j_0$.

In general we write

$$u_{ij}(t) := I + te_{ij} \in U_0(k)$$

for $i < j$ and $t \in k$. If $k < l$, then we see, using (1.1.7), that

$$u_{ij}(t)u_{kl}(t')u_{ij}(-t) = \begin{cases} u_{kl}(t')u_{il}(tt'), & \text{if } j = k; \\ u_{kl}(t')u_{kj}(-tt'), & \text{if } l = i; \\ u_{kl}(t'), & \text{otherwise.} \end{cases} \tag{1.1.8}$$

Write the block-diagonal matrix

$$u_{\lambda'}(y) = \operatorname{diag}(u_{(j_0)}, \dots, u_{(j_r)}), \tag{1.1.9}$$

where $j_0 + \cdots + j_r = n-1$.

Lemma 1.1.7. *If $j_r \geq j_0$ and $v_{n-1} \neq 0$, then Proposition 1.1.4 is true.*

Proof. Note that $v = \prod_{j_0 \leq j \leq n-1} u_{j,n}(v_j)$, a commutative product, and that

$$u_{\lambda'}(y) = u_{n-2,n-1}(y_{n-2,n-1}) \cdots u_{12}(y_{12}).$$

Let $s = j_0 + \cdots + j_{r-1}$. If $s+1 = n - j_r \leq n - j_0$, then we may employ (1.1.8) and a routine induction to show that $t_1, \dots, t_{j_0}$ may be chosen such that

$$u_{1,n-j_0}(t_{j_0}) \cdots u_{j_0,n-1}(t_1)(v, u_{\lambda'}(y))u_{j_0,n-1}(-t_1) \cdots u_{1,n-j_0}(-t_{j_0}) = (v', u_{\lambda'}(y)),$$

where $v'_j = v_j$ for $j \neq j_0$ and $v'_{j_0} = 0$. To finish the proof apply Lemma 1.1.6. □

If we cannot apply Lemma 1.1.7, then we know that either $v_{n-1} = 0$ or $j_r < j_0$.

To complete the proof of Proposition 1.1.4 we shall have to consider conjugations by permutation matrices which belong to $\mathcal{Q}_n(k)$. Obviously a permutation matrix p belongs to $\mathcal{Q}_n(k)$ if and only if p may be identified with $(0, p)$, where $p \in \mathrm{G}_{n-1}(k)$ and $(0,p)(v,g)(0,p^{-1}) = (pv, pgp^{-1})$. Clearly, p is completely determined by the permutation pv of the components of v. Conjugation by p permutes both the rows and the columns of g in the same way as it permutes the components of v.

Lemma 1.1.8. *There exists a cyclic permutation matrix $p \in G_{n-1}(k)$ such that for the block-diagonal matrix $u_{\lambda'}(y)$ (see (1.1.9))*

$$(0,p)(v, u_{\lambda'}(y))(0, p^{-1}) = (v', u_{\lambda''}),$$

where $u_{\lambda''} = \operatorname{diag}(u_{(j_r)}, u_{(j_0)}, \dots, u_{(j_{r-1})})$ and $v'_i = [pv]_i = v_{i+s}$, where $s = j_0 + \cdots + j_{r-1}$ and $i+s$ denotes the residue mod $n-1$ in the set $\{1, \dots, n-1\}$.

Proof. As already observed, p is defined by its action on k_+^{n-1} and we have given this action. Moreover, conjugation by p induces the same cyclic permutation on the rows and the columns of $u_{\lambda'}(y)$. More precisely, the cyclic permutation of the rows of $u_{\lambda'}(y)$, which moves the rows of $u_{\lambda'}(y)$ "down" j_r steps, moves the rows of the last block to the top j_r rows of the transformed matrix; the same permutation of the columns moves them j_r steps to the right and transforms the last j_r columns into the first j_r columns of $pu_{\lambda'}(y)p^{-1}$. This preserves the block-diagonal structure of the matrix. □

To complete the proof of Proposition 1.1.4 it is now enough to observe that if $v_{n-1} = 0$, then the permutation of Lemma 1.1.8 sends this zero to the j_r-th component of v. In this case, since there is also a zero in the j_r, j_r+1-th place of $u_{\lambda''}$, we can apply Lemma 1.1.6. Otherwise, we must repeat the permutation construction of Lemma 1.1.8 either until a zero appears in the component of v at the end of the first block, in which case Lemma 1.1.6 applies, or until the first block is no larger than the last block and a non-zero entry occurs in the $n-1$-th place of v, in which case Lemma 1.1.7 completes the proof. □

Corollary 1.1.9. *Let $u_\lambda(x) \in U_0(k_d)$ with $tr_{k_d|k}(x_{i,i+1}) = y_{i,i+1} \neq 0$ if and only if $x_{i,i+1} \neq 0$. Then the matrix $\mathcal{N}(u_\lambda(x))$ is $U_0(k_d)$-conjugate to $u_\lambda(y) \in U_0(k)$.*

Proof. Obviously the matrix norm preserves the block decomposition of $u_\lambda(x)$. Thus our hypothesis implies that $\mathcal{N}(u_\lambda(x)) \in U_0(k_d)$ is a block-diagonal matrix with the super diagonal entries $y_{i,i+1} = \mathrm{tr}_{k_d|k}(x_{i,i+1})$. By Proposition 1.1.4 it is possible to conjugate the block-diagonal matrix $\mathcal{N}(u_\lambda(x))$ by a block-diagonal upper triangular unipotent matrix and to transform it into a Jordan matrix without changing the super-diagonal entries of $\mathcal{N}(u_\lambda(x))$. □

To prove the implication $(i)_{n-1} \Rightarrow (ii)_n$ we use the injective map $\iota_{n,n-1}$ defined in [HM, Proposition A.3.1] and an obvious generalization $\tilde{\iota}_{n,n-1}$:

$$\begin{aligned} \iota_{n,n-1} &: \mathrm{Irr}(\mathcal{Q}_{n-1}(k)) \to \mathrm{Irr}(\mathcal{Q}_n(k)) \\ \tilde{\iota}_{n,n-1} &: \mathrm{Irr}(\tilde{\mathcal{Q}}_{n-1}(k_d)) \to \mathrm{Irr}(\tilde{\mathcal{Q}}_n(k_d)). \end{aligned} \tag{1.1.10}$$

These character relations imply that

$$\Theta_{\tilde{\rho}_n} = \tilde{\iota}_{n,n-1}(\Theta_{\tilde{\rho}_{n-1}}) \qquad \text{and} \qquad \Theta_{\rho_n(k)} = \iota_{n,n-1}(\Theta_{\rho_{n-1}(k)}).$$

Indeed, $\tilde{\iota}_{n,n-1}$ and $\iota_{n,n-1}$ give induced character relations which we express explicitly as

$$\begin{aligned} \Theta_{\tilde{\rho}_n}(u_\lambda(x)\phi) &= \sum_{0\neq\ell\in L} \psi_1(\ell_{n-1}y_{n-1,n})\Theta_{\tilde{\rho}_{n-1}}(A_\ell u_{\lambda'}(x)\phi A_\ell^{-1}) \\ \Theta_{\rho_n(k)}(u_\lambda(y)) &= \sum_{0\neq\ell\in L} \psi_1(\ell_{n-1}y_{n-1,n})\Theta_{\rho_{n-1}(k)}(A_\ell u_{\lambda'}(y)A_\ell^{-1}), \end{aligned} \tag{1.1.11}$$

where

$$L = \{\ell = (\ell_1, \dots, \ell_{n-1}) \in k^{n-1} \mid \ell_i x_{i,i+1} = 0 \quad (\forall i \in [1, \dots, n-2])\}$$

and $A_\ell \in \mathrm{G}_{n-1}(k)$ denotes any fixed representative with ℓ as its last row. It is important that A_ℓ commutes with ϕ and that $\ell_i x_{i,i+1} = 0$ if and only if $\ell_i y_{i,i+1} = 0$. Applying our hypothesis $(i)_{n-1}$ to the right side of (1.1.11), we see that

$$\begin{aligned}\Theta_{\tilde{\rho}_{n-1}}(A_\ell u_{\lambda'}(x)\phi A_\ell^{-1}) &= \Theta_{\rho_{n-1}(k)}([\mathcal{N}(A_\ell u_{\lambda'}(x)A_\ell^{-1})]) \\ &= \Theta_{\rho_{n-1}(k)}(A_\ell[\mathcal{N}(u_{\lambda'}(x))]A_\ell^{-1}) \\ &= \Theta_{\rho_{n-1}(k)}(A_\ell u_{\lambda'}(y)A_\ell^{-1}),\end{aligned}$$

which implies that the right sides of (1.1.11) are equal.

To complete the proof we have to show that the maps (1.1.10) do give the character relations (1.1.11). For this we recall the characters ψ_1 of k_+ and $\psi_d = \psi_1 \circ \mathrm{tr}_{k_d|k}$ of $k_{d,+}$ (reference). Let k_d^{n-1} denote the space of $n-1 \times 1$ column vectors with components in k_d and $k_d^{n-1^t}$ the transpose space, i. e. the space of row vectors with components in k_d. We use ψ_d to identify $k_d^{n-1^t}$ with the group of characters of the group under addition $k_{d,+}^{n-1}$. Concretely, for $\ell \in k_d^{n-1^t}$ we set

$$\psi_d \circ \ell(v) = \psi_d(\ell v)$$

for $v \in k_d^{n-1}$. Thus we have an isomorphism from the additive group $k_{d,+}^{n-1^t}$

$$\ell \mapsto \psi_d \circ \ell \tag{1.1.12}$$

to the character group $X(k_{d,+}^{n-1})$. We define a right action of $\tilde{\mathrm{G}}_{n-1}(k_d)$ on (1.1.12) by setting

$$(\psi_d \circ \ell)^g := \psi_d \circ (\ell g) \qquad \text{and} \qquad (\psi_d \circ \ell)^\phi := \psi_d \circ \phi^{-1}(\ell).$$

This gives an action of $\tilde{\mathrm{G}}_{n-1}(k_d)$ on $X(k_{d,+}^{n-1})$. We note that all non-trivial characters form one orbit under this action.

Let $e_{n-1} = (0, \dots, 0, 1) \in k_{d,+}^{n-1^t}$. Clearly, $\tilde{\mathcal{Q}}_{n-1}(k_d)$ is the stabilizer of $\psi_d \circ e_{n-1}$ in $\tilde{\mathrm{G}}_{n-1}(k_d)$. In making the injection $\tilde{\iota}_{n,n-1}$ explicit we again make use of the affine realization of $\mathcal{Q}_n(k_d)$. For Θ any irreducible character of $\tilde{\mathcal{Q}}_{n-1}(k_d)$ we set

$$\tilde{\iota}_{n,n-1}(\Theta) := \mathrm{Ind}_{\tilde{\mathcal{Q}}_{n-1,1}(k_d)}^{\tilde{\mathcal{Q}}_n(k_d)}(\psi_d \circ e_{n-1} \cdot \Theta),$$

where

$$\tilde{\mathcal{Q}}_{n-1,1}(k_d) := k_{d,+}^{n-1} \rtimes \tilde{\mathcal{Q}}_{n-1}(k_d) \subset \tilde{\mathcal{Q}}_n(k_d) = k_{d,+}^{n-1} \rtimes \tilde{\mathrm{G}}_{n-1}(k_d)$$

and

$$(\psi_d \circ e_{n-1} \cdot \Theta)((v, g)\phi^i) = \psi_d(e_{n-1}v) \cdot \Theta(g\phi^i).$$

If we take $k_d = k$, i. e. $d = 1$ and ϕ trivial, then we obtain the mapping $\iota_{n,n-1}$ (see (1.1.10)).

Our aim is to prove a formula for the character $\Theta_n := \tilde{\iota}_{n,n-1}(\Theta)$. For this we need a set of representatives R for

$$\tilde{\mathcal{Q}}_{n-1,1}(k_d)\backslash\tilde{\mathcal{Q}}_n(k_d) = \tilde{\mathcal{Q}}_{n-1}(k_d)\backslash\tilde{\mathrm{G}}_{n-1}(k_d). \tag{1.1.13}$$

Lemma 1.1.10. *A complete set of representatives R for (1.1.13) may be given by choosing a matrix $A_\ell \in G_{n-1}(k_d)$ for each non-zero row vector $\ell \in {k_d^{n-1}}^t$ such that ℓ is the last row of A_ℓ.*

Proof. Let $A_\ell, B_\ell \in \mathrm{G}_{n-1}(k_d)$ be any two matrices with the same last row $\ell \neq 0$. Then

$$B_\ell A_\ell^{-1} = I_{n-1} + (B_\ell - A_\ell)A_\ell^{-1};$$

this matrix has e_{n-1} as its last row, as we see from the right side. Therefore, the right coset $\tilde{\mathcal{Q}}_{n-1}(k_d)A_\ell$ depends only upon ℓ and not on the matrix with ℓ as last row. Now let K denote the field extension $k_{d(n-1)}|k_d$ of degree $n-1$ and embed $K \hookrightarrow \mathrm{M}_{n-1}(k_d)$; then certainly $K^\times \hookrightarrow \mathrm{G}_{n-1}(k_d)$. Since K is also closed under addition and contains precisely one element with last row zero, we know that there is a bijection $B_\ell \mapsto \ell$ from $K^\times \subset \mathrm{G}_{n-1}(k_d)$ to the set of non-zero elements of ${k_d^{n-1}}^t$ which assigns each matrix its last row. Since $\tilde{\mathcal{Q}}_{n-1}(k_d) \cap K^\times = (I)$, we have $\tilde{\mathcal{Q}}_{n-1}(k_d) \cdot K^\times = \tilde{\mathrm{G}}_{n-1}(k_d)$. Therefore, the matrices B_ℓ form a system R, which implies that the matrices A_ℓ also form such a system. $\square$

Now, using Frobenius' induced character formula, we find that

$$\Theta_n((v,g)\phi^i) = \sum_{0 \neq \ell \in {k_d^{n-1}}^t} (\psi_d \circ e_{n-1} \cdot \Theta)((0,A_\ell)(v,g)\phi^i(0,A_\ell)^{-1}), \tag{1.1.14}$$

where $\psi_d \circ e_{n-1} \cdot \Theta$ has to be extended by zero from $\tilde{\mathcal{Q}}_{n-1,1}(k_d)$ to $\tilde{\mathcal{Q}}_n(k_d)$. Note that

$$(0,A_\ell)(v,g)\phi^i(0,A_\ell)^{-1} = (A_\ell v, A_\ell g\, {}^{\phi^i}\!A_\ell^{-1})\phi^i \tag{1.1.15}$$

and, moreover:

Lemma 1.1.11. *For any $g \in G_{n-1}(k_d)$*

$$A_\ell g\, {}^{\phi^i}\!A_\ell^{-1} \in \mathcal{Q}_{n-1}(k_d) \iff \ell g = {}^{\phi^i}\ell.$$

Proof. By definition $x \in \mathrm{G}_{n-1}(k_d)$ belongs to $\mathcal{Q}_{n-1}(k_d)$ if and only if the last row of x is $e_{n-1} = (0,\dots,0,1)$, i. e. $e_{n-1}x = e_{n-1}$. Clearly, $e_{n-1}A_\ell g\, {}^{\phi^i}\!A_\ell^{-1} = e_{n-1}$ if and only if $e_{n-1}A_\ell g = e_{n-1}\, {}^{\phi^i}\!A_\ell$, and this means that $\ell g = {}^{\phi^i}\ell$. $\square$

We set

$$L := L(g,i) := \{\ell \in {k_d^{n-1}}^t \mid \ell g = {}^{\phi^i}\ell\}.$$

Then (1.1.14-15) and the fact that $(\psi_d \circ e_{n-1})(A_\ell v) = \psi_d(\ell v)$ imply that

$$\Theta_n((v,g)\phi^i) = \sum_{0 \neq \ell \in L(g,i)} \psi_d(\ell v) \cdot \Theta(A_\ell g\, {}^{\phi^i}\!A_\ell^{-1}\phi^i). \tag{1.1.16}$$

In the special case $d = 1$ and $i = 0$ we obtain a formula for $\iota_{n,n-1}(\Theta)$, assuming in this case that Θ is a character of $\mathcal{Q}_{n-1}(k)$.

Now we consider (1.1.16) for the argument $u_\lambda(x)\phi$ for $u_\lambda(x) \in U_0(k_d)$. Since $u_\lambda(x) = ([0,\dots,0,x_{n-1,n}]^t, u_{\lambda'}(x))$, where $\lambda' = (\lambda_1,\dots,\lambda_{p-1},\lambda_p - 1)$ is a partition of $n-1$, we have to sum over the set

$$L(u_{\lambda'}(x),1) = \{\ell \in {k_d^{n-1}}^t \mid \ell u_{\lambda'}(x) = {}^{\phi}\ell\}.$$

The condition that $\ell u_{\lambda'}(x) = {}^{\phi}\ell$ may be expressed in the form

$$(\ell_1, \ell_1 x_{1,2} + \ell_2, \dots, \ell_{n-2}x_{n-2,n-1} + \ell_{n-1}) = ({}^{\phi}\ell_1, \dots, {}^{\phi}\ell_{n-1}).$$

Next it is necessary to verify that

$$L(u_{\lambda'}(x),1) = \{\ell = (\ell_1,\dots,\ell_{n-1}) \in k^{n-1} \mid \ell_i x_{i,i+1} = 0 \quad \forall i, 1 \le i \le n-2\}.$$

That the given properties of the coordinates ℓ_i are both necessary and sufficient may be checked by continuing the following line of argument. For $i = 1$ we must have $\ell_1 = {}^{\phi}\ell_1$ and $\ell_1 x_{12} = {}^{\phi}\ell_2 - \ell_2$; these conditions are satisfied if and only if $\ell_1 \in k$ and $\mathrm{tr}_{k_d|k}(\ell_1 x_{12}) = 0$. To prove this for all i the reader should formulate an inductive argument.

Since the row vectors ℓ have components in k, we may choose representatives $A_\ell \in \mathrm{G}_{n-1}(k)$ too. Now from (1.1.16) and the equalities

$$\begin{aligned}\psi_d(\ell[0,\dots,0,x_{n-1,n}]^t) &= \psi_d(\ell_{n-1}x_{n-1,n})\\ &= \psi_1(\ell_{n-1}\mathrm{tr}_{k_d|k}(x_{n-1,n}))\\ &= \psi_1(\ell_{n-1}y_{n-1,n})\end{aligned}$$

we obtain the first formula of (1.1.11). Taking $d = 1$, $i = 0$, and $u_\lambda(x) \in U_0(k)$, we obtain, as already noted, the second formula $(1.1.11)_2$.

For the next Corollary 1.1.12 we introduce the notation

$$\phi_k(t) = \prod_{j=1}^{k}(1-t^j) \quad (k \ge 1) \quad \text{and} \quad \phi_0(t) = 1, \tag{1.1.17}$$

which we shall also use frequently in §2.

Corollary 1.1.12. *Let λ be a partition of n consisting of p parts. Let $u_\lambda(x) \in U_0(k_d)$ be a special Jordan matrix associated to λ and $u_\lambda := u_\lambda(y) \in U_0(k)$ any Jordan matrix corresponding to λ. Then*

$$\Theta_{\tilde\rho_n(k_d)}(u_\lambda(x)\phi) = \Theta_{\rho_n(k)}(u_\lambda) = (-1)^{n-1}\phi_{p-1}(q). \tag{1.1.18}$$

Proof. Proposition 1.1.1 implies that it is sufficient to prove the second equality. We argue by induction on n, the case $n = 1$ being trivial. Assume that the formula is true for all integers less than $n > 1$. We are going to use $(1.1.11)_2$:

$$\Theta_n(u_\lambda(y)) = \sum_{0\ne\ell\in L} \psi_1(\ell_{n-1}y_{n-1,n})\Theta_{n-1}(A_\ell u_{\lambda'}(y)A_\ell^{-1}),$$

where $\lambda = (\lambda_1, \dots, \lambda_p)$ is a non-ordered partition of n, $\lambda' = (\lambda_1, \dots, \lambda_p - 1)$ is a non-ordered partition of $n-1$, and

$$L = \{(\ell_1, \dots, \ell_{n-1}) \in k_{n-1} \mid \ell_i y_{i,i+1} = 0,\ \forall i, 1 \leq i \leq n-2\}.$$

In this case, $A_\ell u_{\lambda'}(y) A_\ell^{-1} \in \mathcal{Q}_{n-1}(k)$ is a unipotent element, hence, according to Proposition 1.1.4, it is $\mathcal{Q}_{n-1}(k)$ conjugate to a Jordan matrix u_μ for some partition μ of $n-1$. In this case, $u_{\lambda'}(y)$ and u_μ are $\mathrm{G}_{n-1}(k)$ conjugate, which implies that the partitions λ' and μ have the same number of parts. Therefore, by the induction hypothesis, we have

$$\Theta_{n-1}(A_\ell u_{\lambda'}(y) A_\ell^{-1}) = \Theta_{n-1}(u_\mu) = \Theta_{n-1}(u_{\lambda'}(y)).$$

Therefore,

$$\Theta_n(u_\lambda) = \left\{ \sum_{0 \neq \ell \in L} \psi_1(\ell_{n-1} y_{n-1,n}) \right\} \cdot \Theta_{n-1}(u_{\lambda'}(y)).$$

Note that the set L is a k-vector space. If $\lambda_p = 1$, the dimension of L is $p-1$ and $y_{n-1,n} = 0$, so the character $\ell \mapsto \psi_1(\ell_{n-1} y_{n-1,n})$ is trivial and the value of the sum within braces is $q^{p-1} - 1$. If $\lambda_p > 1$, then $y_{n-1,n} \neq 0$, the character is non-trivial and the value of the sum is -1. Therefore,

$$\Theta_n(u_\lambda) = \begin{cases} -(1 - q^{p-1})\Theta_{n-1}(u_{\lambda'}(y)), & \text{if } \lambda_p = 1; \\ -\Theta_{n-1}(u_{\lambda'}(y)), & \text{if } \lambda_p \neq 1. \end{cases}$$

By substitution and application of the induction hypothesis we obtain the stated result. $\square$

Remark 1.2. Corollary 1.1.12 implies the fact, which we made use of in the proof, that the character Θ_n of $\mathcal{Q}_n(k)$ is constant on $\mathrm{G}_n(k)$-conjugacy classes. This fact also follows from the simple observation that the restriction of a cuspidal representation of $\mathrm{G}_n(k)$ to $\mathcal{Q}_n(k)$ must be a multiple of $\rho_n(k)$. This observation is weaker than S. I. Gelfand's Theorem [G1,G2] that the multiple is in fact one. Gelfand's Theorem follows from Corollary 1.1.12 by comparing our character formula (1.1.18) to [GR, p. 431, the third formula].

2. The Green's Parameter of the Descent of a Cuspidal Representation

The purpose of this section is to present the proof of Theorem 2.

For all $l|n$ we set

$$F_l := F_l(\chi_{dn}) := \{\eta \in X(k_{dn}^\times) \mid \eta|_{k_{dl}^\times} \underset{k_d}{\sim} \chi_{dn}|_{k_{dl}^\times}\}, \tag{2.1}$$

where "$\underset{k_d}{\sim}$" means that the restriction $\eta|_{k_{dl}^\times}$ is $\mathrm{Gal}(k_{dl}|k_d)$-conjugate to $\chi_{dn}|_{k_{dl}^\times}$. For instance, in the extreme cases, F_1 is the set of all extensions to $k_{dn}^\times$ of the restriction $\chi_{dn}|_{k_d^\times}$ and $F_n = [\chi_{dn}]$, the $\mathrm{Gal}(k_{dn}|k_d)$-orbit of χ_{dn}.

For any $l|n$ let $Z(\chi_{dn} : k_{dl}|k_d)$ denote the stabilizer of $\chi_{dn}|_{k_{dl}^\times}$ in $\mathrm{Gal}(k_{dl}|k_d)$ and let

$$z(\chi_{dn} : k_{dl}|k_d) := |Z(\chi_{dn} : k_{dl}|k_d)|, \tag{2.2}$$

the order of $Z(\chi_{dn} : k_{dl}|k_d)$. If $\chi_{dn}|_{k_{dl}^\times}$ is a regular $k_{dl}|k_d$-character, then $z(\chi_{dn} : k_{dl}|k_d) = 1$.

Assume that $[\chi_{dn}]$ is a $\mathrm{Gal}(k_{dn}|k_d)$-orbit of k_d-regular characters of $k_{dn}^\times$. Let $\Pi := \Pi_{\chi_{dn}}$ denote the corresponding cuspidal representation of $\mathrm{G}_n(k_d)$. Let $\Theta := \Theta_\Pi$ denote the character of Π and let $\Theta_\Gamma := \Theta|_{\Gamma(k_d)}$ denote the restriction of Θ to $\Gamma(k_d)$.

Proposition 2.1.

$$\Theta_\Gamma = (-1)^{n-1} \sum_{l|n} \Lambda_l, \tag{2.3}$$

where

$$\Lambda_l = m_l(q^d) z(\chi_{dn} : k_{dl}|k_d) \sum_{\eta \in F_l} \eta \tag{2.4}$$

with $m_l(X) \in \mathbb{Z}[X]$, up to sign, a monic polynomial.

Proof. As Proposition 2.1 does not depend upon the extension $k_d|k$, we take $d = 1$ and $\chi_n \in X(k_n^\times)$ regular over k and prove the Proposition for Π a cuspidal representation of $\mathrm{G}_n(k)$. For $x \in \Gamma(k) \subset \mathrm{G}_n(k)$, where $\Gamma(k) \cong k_n^\times$, and $l = [k(x) : k]$ we obtain from [GR, p. 431, the third equation]

$$\begin{aligned} \Theta_{\chi_n}(x) := \Theta_{\Pi_{\chi_n}}(x) &= (-1)^{n-1} \phi_{\frac{n}{l}-1}(q^l) \sum_{j=0}^{l-1} \chi_n^{q^j}(x) \\ &= (-1)^{n-1} \phi_{\frac{n}{l}-1}(q^l) z(\chi_n : k_l|k) \sum_{\eta \in [\chi_n|_{k_l^\times}]} \eta, \end{aligned} \tag{2.5}$$

where $[\chi_n|_{k_l^\times}]$ denotes the $\mathrm{Gal}(k_l|k)$-orbit of $\chi_n|_{k_l^\times}$ and the polynomials $\phi_k(t)$ were defined in (1.1.17). If the restriction $\chi_n|_{k_l^\times}$ is a regular character of $k_l^\times$, then $z(\chi_n : k_l|k) = 1$. As examples, let us again cite the extreme cases

$$\Theta_{\chi_n}(x) = \begin{cases} (-1)^{n-1} \sum_{\eta \in [\chi_n]} \eta(x), & \text{if } l = n, \\ (-1)^{n-1} \phi_{n-1}(q) \chi_n(x), & \text{if } l = 1. \end{cases}$$

Using (2.1) and (2.2) with $d = 1$, we introduce the “Ansatz”

$$\Theta_\Gamma = (-1)^{n-1} \sum_{l|n} \Lambda_l, \qquad \Lambda_l = m_l(q) z(\chi_n : k_l|k) \sum_{\eta \in F_l} \eta. \tag{2.6}$$

Then for $x \in \Gamma$ such that $k(x) \cong k_r$ we have

$$\Theta_\Gamma(x) = (-1)^{n-1} \sum_{l:r|l|n} \Lambda_l(x). \tag{2.7}$$

If $r \nmid l$, i. e. if $x \notin k_l$, then

$$\begin{aligned} z(\chi_n : k_l|k) \sum_{\eta \in F_l} \eta(x) &= \sum_{j=0}^{l-1} \chi_n^{q^j}(x) \sum_{\eta \in X(k_n^\times/k_l^\times)} \eta^{q^j}(x) \\ &= \sum_{j=0}^{l-1} \chi_n^{q^j}(x) \sum_{\eta \in X(k_n^\times/k_l^\times)} \eta(x) \\ &= 0. \end{aligned} \tag{2.8}$$

If $r|l$, then $x \in k_l^\times$ and $\sum_{\eta \in X(k_n^\times/k_l^\times)} \eta(x) = [k_n^\times : k_l^\times]$. In this case,

$$z(\chi_n : k_l|k) \sum_{\eta \in F_l} \eta(x) = [k_n^\times : k_l^\times] \sum_{j=0}^{l-1} \chi_n^{q^j}(x) = [k_n^\times : k_l^\times] \frac{l}{r} \sum_{j=0}^{r-1} \chi_n^{q^j}(x). \tag{2.9}$$

From (2.6)-(2.9) we obtain for $k(x) \cong k_r^\times$ that

$$\Theta_\Gamma(x) = (-1)^{n-1} \sum_{l:r|l|n} m_l(q)[k_n^\times : k_l^\times] \frac{l}{r} \sum_{j=0}^{r-1} \chi_n^{q^j}(x).$$

Comparing this expression with (2.5) for $k(x) \cong k_r$ we see that (2.6) is valid provided that

$$\sum_{l:r|l|n} m_l(q)[k_n^\times : k_l^\times] \frac{l}{r} = \phi_{\frac{n}{r}-1}(q^r). \tag{2.10}$$

Setting $l' = n/l$ and $r' = n/r$, we rewrite this formula as

$$\sum_{l':l'|r'} m_{n/l'}(q)[k_n^\times : k_{n/l'}^\times] \frac{n}{l'} = \frac{n}{r'} \phi_{r'-1}(q^{n/r'}), \tag{2.11}$$

apply Möbius' inversion formula, and obtain

$$m_{n/r'}(q)[k_n^\times : k_{n/r'}^\times] \frac{n}{r'} = \sum_{l'|r'} \mu(\frac{r'}{l'}) \frac{n}{l'} \phi_{l'-1}(q^{n/l'})$$

or

$$m_r(q) = \frac{1}{[k_n^\times : k_r^\times]} \sum_{l:r|l|n} \mu(\frac{l}{r}) \frac{l}{r} \phi_{\frac{n}{l}-1}(q^l). \tag{2.12}$$

Substituting (2.12) into (2.6), we satisfy the Ansatz and obtain the expression for Θ_Γ asserted in the Proposition. Because of the uniqueness of the representation of any function on Γ as a linear combination of characters, we conclude that

$$\Theta_{\chi_n}|_\Gamma = \Theta_\Gamma,$$

if for $r|n$ we take $m_r(q)$ as in (2.12).

We must show that $m_r(q)$ is the value of an integer polynomial which is monic up to sign. For $r|n$ we define

$$m_{r,n}(X) := \frac{X^r - 1}{X^n - 1} \sum_{l:r|l|n} \mu(\frac{l}{r})\frac{l}{r}\phi_{\frac{n}{l}-1}(X^l), \tag{2.13}$$

which is certainly a quotient of polynomials with integer coefficients satisfying the relation $m_{r,n}(q) = m_r(q)$, by (2.12). Since (2.13) implies that

$$m_{r,n}(X) = m_{1,\frac{n}{r}}(X^r), \tag{2.14}$$

it is enough to show that $m_{1,n}(X)$ is an integer polynomial which has leading coefficient ± 1. Using (2.13) and noting that $\phi_{n-1}(X)$ has larger degree than $\phi_{\frac{n}{l}-1}(X^l)$ for every $l > 1$, we see that $m_{1,n}(X)$ is, up to sign, a quotient of monic integer polynomials. Thus if it is a polynomial, then it is certainly, up to sign, a monic polynomial with integer coefficients. To show that $m_{1,n}(X)$ is a polynomial, and not just a quotient of integer polynomials, it follows from the division algorithm that it is enough to show that $m_1(q) = m_{1,n}(q)$ is an integer for infinitely many q. But, from (2.3) and (2.4), we have

$$\Theta_{\chi_n}|_\Gamma = (-1)^{n-1} \sum_{l|n} m_l(q) z(\chi_n : k_l|k) \sum_{\eta \in F_l} \eta, \tag{2.15}$$

and for $q \geq 3$ characters η occur in F_1 which do not belong to F_l for any $l > 1$, since for $q \geq 3$

$$\frac{q^n - 1}{q - 1} > \sum_{l=2}^{\infty} l \frac{q^n - 1}{q^l - 1}. \qquad \square$$

Examples.

(i) For fixed n we usually write $m_l(X)$ instead of $m_{l,n}(X)$, hence $m_n(X) = m_{n,n}(X) = 1$. Thus, if $\chi_{dn} \in X(k_{dn}^\times)$ is both k_d-regular (§0.4) and $\langle\phi\rangle$-invariant (§0.2), then

$$\Lambda_n = \sum_{\eta \in [\chi_{dn}]} \eta = \sum_{\eta' \in [\chi_n]} \eta' \circ \mathrm{N}_{k_{dn}|k_n}.$$

We have $z(\chi_{dn} : k_{dn}|k_d) = 1$ and $F_n = [\chi_{dn}] = [\chi_n \circ \mathrm{N}_{k_{dn}|k_n}]$.

(ii) Note that $m_{1,2}(X) = -1$ and $m_{1,3}(X) = X - 2$. These examples imply that $m_{3,6}(X) = -1$ and $m_{2,6}(X) = X^2 - 2$.

Return to $\Pi = \Pi_{\chi_{dn}}$ a $\langle\phi\rangle$-invariant cuspidal representation of $\mathrm{G}_n(k_d)$, i. e. $(d,n)=1$, and $\chi_{dn} = \chi_n \circ \mathrm{N}_{k_{dn}|k_n}$ for some k-regular $\chi_n \in X(k_n^\times)$. Set

$$\tilde{\Gamma} := \Gamma(k_d) \rtimes \langle\phi\rangle \subset \mathrm{G}_n(k_d) \rtimes \langle\phi\rangle. \tag{2.16}$$

To prove Theorem 2 we shall have to study the character

$$\Theta_{\tilde{\Gamma}} := \Theta_{\tilde{\Pi}}|_{\tilde{\Gamma}}. \tag{2.17}$$

For all $i = 0, \dots, d-1$ define

$$E_{l,i} := \{\eta \in X(\Gamma(k_d))^{\langle\phi^i\rangle} \mid \eta|_{k_{dl}^\times} \underset{k_d}{\sim} \chi_{dn,i}|_{k_{dl}^\times}\} = F_l^{\langle\phi^i\rangle}(\chi_{dn,i}), \tag{2.18}$$

where

$$\chi_{dn,i} := \chi_{(d,i)n,i} \circ \mathrm{N}_{k_{dn}|k_{(d,i)n}}. \tag{2.19}$$

Also set

$$\tilde{\Lambda}_l(\gamma\phi^i) := m_l(q^{(d,i)}) z(\chi_{dn,i} : k_{dl}|k_d) \sum_{\eta \in E_{l,i}} \eta(\gamma) \qquad (\gamma \in \Gamma). \tag{2.20}$$

Write

$$z_{l,i} := z(\chi_{dn,i} : k_{dl}|k_d) = z(\chi_{n,i} : k_l|k), \tag{2.21}$$

the last equality being a consequence of Theorem 1.

For $i = 0$ Proposition 2.1 implies that

$$\tilde{\Lambda}_l(\gamma) = m_l(q^d) z_{l,0} \sum_{\eta \in F_l} \eta(\gamma) = \Lambda_l(\gamma), \tag{2.22}$$

since $E_{l,0} = F_l$ and $z_{l,0} = z(\chi_{dn} : k_{dl}|k_d)$. Thus $\tilde{\Lambda}_l$ is a central function on $\tilde{\Gamma}$ which extends Λ_l.

Proposition 2.2.

$$\Theta_{\tilde{\Gamma}} = (-1)^{n-1} \sum_{l|n} \tilde{\Lambda}_l. \tag{2.23}$$

Proof. Proposition 2.1 applied to the cuspidal representation $j_i(\Pi)$ of $\mathrm{G}_n(k_{(d,i)})$ implies that

$$\begin{aligned} \Theta_{\tilde{\Gamma}}(\gamma\phi^i) &= \Theta_{j_i(\Pi)}(\mathcal{N}_i\gamma) \\ &= (-1)^{n-1} \sum_{l|n} \Lambda_l(j_i(\Pi))(\mathcal{N}_i\gamma), \end{aligned}$$

where

$$\Lambda_l(j_i(\Pi)) = m_l(q^{(d,i)}) z_{l,i} \sum_{\eta' \in F_l(\chi_{(d,i)n,i})} \eta',$$

$F_l(\chi_{(d,i)n,i}) := \{\eta' \in X(k^\times_{(d,i)n}) \mid \eta'|_{k^\times_{(d,i)l}} \underset{k_{(d,i)}}{\sim} \chi_{(d,i)n,i}|_{k^\times_{(d,i)l}}\}$. Thus,

$$\Lambda_l(j_i(\Pi))(\mathcal{N}_i\gamma) = m_l(q^{(d,i)}) z_{l,i} \sum_{\eta' \in F_l(\chi_{(d,i)n,i})} \eta' \circ \mathrm{N}_{k_{dn}|k_{(d,i)n}}(\gamma)$$

for $\gamma \in \Gamma(k_d)$. We have to show that $F_l(\chi_{(d,i)n,i}) \circ \mathrm{N}_{k_{dn}|k_{(d,i)n}} = E_{l,i}$. The inclusion $\supseteq$ follows from the fact that, by definition, $\eta \in E_{l,i}$ factors through $\mathrm{N}_{k_{dn}|k_{(d,i)n}}$; thus we have an injective mapping $\eta \mapsto \eta'$, which sends $E_{l,i}$ to $F_l(\chi_{(d,i)n,i})$. To see that the mapping is also surjective consider $\eta' \in F_l(\chi_{(d,i)n,i})$, let $\eta = \eta' \circ \mathrm{N}_{k_{dn}|k_{(d,i)n}}$, and let $S = \{\gamma \in k^\times_{dn} \mid \mathrm{N}_{k_{dn}|k_{(d,i)n}}(\gamma) \in k_{(d,i)l}\}$. Then $\eta|_S$ is $\mathrm{Gal}(k_{dl}|k_d)$-conjugate to $\chi_{dn,i}|_S$, where $\chi_{dn,i} = \chi_{(d,i)n,i} \circ \mathrm{N}_{k_{dn}|k_{(d,i)n}}$. If $\gamma \in S$, then $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in k^\times_{dl}$ and $\mathrm{N}_{k_{dn}|k_{(d,i)n}}(\gamma_2) = 1$. Thus, $\eta|_S \underset{k_d}{\sim} \chi_{dn,i}|_S$ if and only if $\eta|_{k^\times_{dl}} \underset{k_d}{\sim} \chi_{dn,i}|_{k^\times_{dl}}$, which means that $\eta \in E_{l,i}$. $\square$

Besides the true character $\Theta_{\tilde{\Gamma}}$ and its component central functions $\tilde{\Lambda}_l$ we also consider for all factors $l \mid n$ the central functions Λ'_l on $\tilde{\Gamma}$ which are defined by the formulas

$$\Lambda'_l(\gamma\phi^i) := m_l(q^{(d,i)}) z_{l,0} \sum_{\eta \in F_l^{\langle\phi^i\rangle}} \eta(\gamma), \tag{2.24}$$

where $F_l^{\langle\phi^i\rangle}$ denotes the set of $\langle\phi^i\rangle$-invariant characters contained in F_l. The central function Λ'_l also gives an extension of Λ_l to $\tilde{\Gamma}$. We want to compare $\tilde{\Lambda}_l$ and Λ'_l. With the notation of (2.19) Theorem 2 asserts that $[\chi_{dn,i}] = [\chi_{dn,0}]$ for all $i = 0, \ldots, d-1$. This is true if and only if $E_{l,i} = F_l^{\langle\phi^i\rangle}$, $z_{l,i} = z_{l,0}$, and $\tilde{\Lambda}_l = \Lambda'_l$ for all $l \mid n$. In order to prove that $\tilde{\Lambda}_l = \Lambda'_l$ for all l we shall compute the Fourier expansion of $\tilde{\Lambda}_l - \Lambda'_l$ and show that Λ'_l is for all l a virtual character. Putting this together with (2.23) we shall arrive, after some complications, at a proof that $\tilde{\Lambda}_l = \Lambda'_l$ for all l and therefore that Theorem 2 is true too.

We begin with a description of the irreducible characters of $\tilde{\Gamma}$. For any $\xi \in X(\Gamma(k_d))$ let H_ξ denote the stabilizer in $\langle\phi\rangle$ of ξ. Let $\lambda \in X(H_\xi)$ and regard λ as a character of $\Gamma(k_d) \rtimes H_\xi$ by inflation. Let $\tilde{\xi}$ denote the extension of ξ to $\Gamma(k_d) \rtimes H_\xi$ such that $\tilde{\xi}|_{H_\xi} = 1$. Then the character

$$\theta_{\xi,\lambda} := \mathrm{Ind}^{\tilde{\Gamma}}_{\Gamma(k_d)\rtimes H_\xi} \tilde{\xi}\lambda \tag{2.25}$$

is an irreducible character of $\tilde{\Gamma}$. Every irreducible character of $\tilde{\Gamma}$ comes from a coset representative $\xi \in X(\Gamma(k_d))/\langle\phi\rangle$ and choice of $\lambda \in X(H_\xi)$ via the correspondence $(\xi, \lambda) \mapsto \theta_{\xi,\lambda}$ and it is easily seen that this correspondence is bijective. Indeed Frobenius' induced character formula implies that

$$\theta_{\xi,\lambda}(\gamma\phi^i) = \begin{cases} 0, \text{ if } \phi^i \notin H_\xi, \\ S_\xi(\gamma)\lambda(\phi^i), \text{ if } \phi^i \in H_\xi, \end{cases} \tag{2.26}$$

where S_ξ denotes the sum over the $\langle\phi\rangle$-orbit of ξ.

For any finite group Y we write $(,)_Y$ for the normalized inner product on Y.

Proposition 2.3. *With respect to the normalized inner product on* $\tilde{\Gamma}$

$$(\tilde{\Lambda}_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = \sum_{\{i|\,\xi\in E_{l,i}\}} \frac{m_l(q^{(d,i)})}{|H_\xi|} z_{l,i}\lambda^{-1}(\phi^i) \tag{2.27}$$

and

$$(\Lambda'_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = \sum_{\{i|\,\xi\in F_l^{\langle\phi^i\rangle}\}} \frac{m_l(q^{(d,i)})}{|H_\xi|} z_{l,0}\lambda^{-1}(\phi^i). \tag{2.28}$$

Proof. We shall prove only the first assertion as the proof of the second is similar. For the normalized inner product we have

$$(\tilde{\Lambda}_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = \frac{1}{|\tilde{\Gamma}|}\sum_{x\in\tilde{\Gamma}} \tilde{\Lambda}_l(x)\overline{\theta_{\xi,\lambda}(x)} = \frac{1}{|\tilde{\Gamma}|}\sum_{i=0}^{d-1}\sum_{\gamma\in\Gamma(k_d)} \tilde{\Lambda}_l(\gamma\phi^i)\overline{\theta_{\xi,\lambda}(\gamma\phi^i)}.$$

If $\phi^i \notin H_\xi$, then $\theta_{\xi,\lambda}(\gamma\phi^i) = 0$, which implies that

$$(\tilde{\Lambda}_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = \frac{1}{|\tilde{\Gamma}|}\sum_{\{i|\phi^i\in H_\xi\}}\sum_{\gamma\in\Gamma(k_d)} \tilde{\Lambda}_l(\gamma\phi^i)\overline{S_\xi(\gamma)\lambda(\phi^i)}$$

$$= \frac{1}{d}\sum_{\{i|\phi^i\in H_\xi\}} m_l(q^{(d,i)})z_{l,i}\lambda^{-1}(\phi^i)\cdot\frac{1}{|\Gamma(k_d)|}\sum_{\eta\in E_{l,i}}\sum_{\gamma\in\Gamma(k_d)} \eta(\gamma)S_{\xi^{-1}}(\gamma).$$

If $\xi \in E_{l,i}$, then the whole orbit $\langle\phi\rangle\cdot\xi \subset E_{l,i}$, since $\langle\phi\rangle$ stabilizes $E_{l,i}$. It follows that

$$\frac{1}{|\Gamma(k_d)|}\sum_{\eta\in E_{l,i}}\sum_{\gamma\in\Gamma(k_d)} \eta(\gamma)S_{\xi^{-1}}(\gamma) = |\langle\phi\rangle\cdot\xi| = [\langle\phi\rangle : H_\xi] = \frac{d}{|H_\xi|},$$

which implies the assertion of the Proposition in the case $\xi \in E_{l,i}$. Clearly, the orbit $\langle\phi\rangle\cdot\xi$ is orthogonal to η for all $\eta \in E_{l,i}$ if $\xi \notin E_{l,i}$. The assertion of the Proposition is thus true in this case too. □

Proposition 2.4. *The function* Λ'_l *is for any* $l \mid n$ *a virtual character of* $\tilde{\Gamma}$.

Proof. We set up the "Ansatz"

$$\Lambda'_l = z_{l,0}\sum_H \sum_{\{\eta\in F_l/\langle\phi\rangle|\ H_\eta\supseteq H\}} m(\eta, H)\mathrm{Ind}_{\Gamma(k_d)\rtimes H}^{\tilde{\Gamma}}\tilde{\eta}, \tag{2.29}$$

in which H is summed over the subgroups of $\langle\phi\rangle$, H_η denotes the stabilizer of η in $\langle\phi\rangle$, and $\tilde{\eta}$ denotes the extension by the trivial character of H of η to $\Gamma(k_d)\rtimes H$. We want to show that there exist integers $m(\eta, H)$ such that (2.29) is satisfied. For this we compute the character of Λ'_l by using the right side of (2.29). Let $(\mathrm{Ind}_{\Gamma(k_d)\rtimes H}^{\tilde{\Gamma}}\tilde{\eta})(\gamma\phi^i)$ denote the character value. We have

$$(\mathrm{Ind}_{\Gamma(k_d)\rtimes H}^{\tilde{\Gamma}}\tilde{\eta})(\gamma\phi^i) = \begin{cases} 0, & \text{if } \phi^i \notin H, \\ [H_\eta : H]S_\eta(\gamma), & \text{if } \phi^i \in H. \end{cases}$$

The right side of (2.29) has at $\gamma\phi^i$ the value

$$z_{l,0} \sum_{\{H|\phi^i\in H\}} \sum_{\{\eta\in F_l/\langle\phi\rangle \mid H_\eta\supset H\}} m(\eta,H)[H_\eta:H]S_\eta(\gamma).$$

We note that $m(\eta,H)=m(\eta',H)$ if η and η' are in the same $\langle\phi\rangle$-orbit, so we may rewrite this in the form

$$z_{l,0} \sum_{\{H|\phi^i\in H\}} \sum_{\{\eta\in F_l|H_\eta\supseteq H\}} m(\eta,H)[H_\eta:H]\eta(\gamma).$$

From (2.24) it follows that

$$\sum_{\{H|\ H_\eta\supseteq H\supseteq\langle\phi^i\rangle\}} m(\eta,H)[H_\eta:H]=m_l(q^{(d,i)}). \tag{2.30}$$

Set $m(\eta,H)=m(H_\eta,H)$ with $H_\eta\supseteq H$. Then

$$(d,i)=[\langle\phi\rangle:\langle\phi^i\rangle]=[\langle\phi\rangle:H_\eta][H_\eta:\langle\phi^i\rangle]$$

for $\phi^i\in H_\eta$. Set $t=[H_\eta:H]$, $t_0=[H_\eta:\langle\phi^i\rangle]$, $t_\eta=[\langle\phi\rangle:H_\eta]$, and $m(\eta,H)=m(H_\eta,H)=m(H_\eta,t)$. Since the groups H and H_η are cyclic, t parameterizes the set $\{H|H\subseteq H_\eta\}$. We rewrite (2.30) as

$$\sum_{t|t_0} m(H_\eta,t)t=m_l(q^{t_\eta t_0}). \tag{2.31}$$

Now the Möbius inversion formula yields

$$m(H_\eta,t_0)=\frac{1}{t_0}\sum_{t|t_0}\mu(\frac{t_0}{t})m_l(q^{t_\eta t}) \tag{2.32}$$

for any factor $t_0|\ |H_\eta|$. If $t_0=1$, then $m(H_\eta,1)=m_l(q^{t_\eta})\in\mathbb{Z}$. For $t_0\geq 1$ we recall that $m_l(X)=\sum_\nu a_\nu X^\nu\in\mathbb{Z}[X]$. The right side of (2.32) becomes

$$\frac{1}{t_0}\sum_{t|t_0}\mu(\frac{t_0}{t})\sum_\nu a_\nu q_1^{\nu t}=\sum_\nu a_\nu(\frac{1}{t_0}\sum_{t|t_0}\mu(\frac{t_0}{t})q_1^{\nu t}), \tag{2.33}$$

where $q_1=q^{t_\eta}$. Since $\sum_{t|t_0}\mu(\frac{t_0}{t})=0$, the term for $\nu=0$ is zero. Assume $\nu>0$. [LN, Theorem 3.25] tells us that $\frac{1}{t_0}\sum_{t|t_0}\mu(\frac{t_0}{t})q_1^{\nu t}$ equals the number of irreducible polynomials of degree t_0 in $\mathbb{F}_{q_1^\nu}[X]$. Thus (2.32) and (2.33) imply that $m(H_\eta,t_0)$ is an integer. □

Remark 2.1. Since $E_{1,i}=F_1^{\langle\phi^i\rangle}$, it follows that $\tilde{\Lambda}_1=\Lambda_1'$. Therefore, $\tilde{\Lambda}_1$ is also a virtual character.

We continue the proof of Theorem 2 via an induction on the number of prime factors of d, the case $d=1$ being trivial. We take as our induction hypothesis that, for $(d,i)>1$,

$$\chi_{dn,i}=\chi_{n,i}\circ \mathrm{N}_{k_{dn}|k_{(d,i)n}}\in[\chi_{dn}]. \tag{2.34}$$

To prove Theorem 2 it suffices to show that (2.34) also holds when $(d,i)=1$. Clearly, (2.34) holds for i if and only if $E_{l,i}=F_l^{\langle\phi^i\rangle}$ for all $l\mid n$. In particular, it is easy to see that if $E_{l,i}=F_l^{\langle\phi^i\rangle}$ and if $l'|l$, then $E_{l',i}=F_{l'}^{\langle\phi^i\rangle}$ too.

Proposition 2.5. *Let $\xi \in X(\Gamma(k_d))$ and $\lambda \in X(H_\xi)$, where H_ξ denotes the stabilizer in $\langle\phi\rangle$ of ξ.*

(i) If $\xi \notin X(\Gamma(k_d))^{\langle\phi\rangle}$, then $(\tilde{\Lambda}_l - \Lambda'_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = 0$.
Assume that $\xi \in X(\Gamma)^{\langle\phi\rangle}$. Let $\sum'_i$ denote the sum over all i, $0 \le i < d$, such that $(d,i) = 1$.

(ii) If $\xi \notin F_l^{\langle\phi\rangle}$, then

$$(\tilde{\Lambda}_l - \Lambda'_l, \tilde{\xi}\lambda)_{\tilde{\Gamma}} = \sum\nolimits'_{\{i|\,\xi\in E_{l,i}\}} \frac{m_l(q)}{d} z_{l,i}\lambda^{-1}(\phi^i). \tag{2.35}$$

(iii) If $\xi \in F_l^{\langle\phi\rangle}$, then

$$(\tilde{\Lambda}_l - \Lambda'_l, \tilde{\xi}\lambda)_{\tilde{\Gamma}} = -\sum\nolimits'_{\{i|\,\xi\notin E_{l,i}\}} \frac{m_l(q)}{d} z_{l,0}\lambda^{-1}(\phi^i). \tag{2.36}$$

Proof. By Proposition 2.3 we have

$$(\tilde{\Lambda}_l - \Lambda'_l, \theta_{\xi,\lambda})_{\tilde{\Gamma}} = \sum_{\{i|\,\xi\in E_{l,i}\}} \frac{m_l(q^{(d,i)})}{|H_\xi|} z_{l,i}\lambda^{-1}(\phi^i) - \sum_{\{i|\,\xi\in F_l^{\langle\phi^i\rangle}\}} \frac{m_l(q^{(d,i)})}{|H_\xi|} z_{l,0}\lambda^{-1}(\phi^i).$$

From the induction hypothesis, which asserts that $E_{l,i} = F_l^{\langle\phi^i\rangle}$ whenever $(d,i) > 1$, it follows that we need sum only over i such that $(d,i) = 1$ because the other terms cancel each other. Moreover, since $E_{l,i} \subset X(\Gamma(k_d))^{\langle\phi\rangle}$ and $F_l^{\langle\phi^i\rangle} \subset X(\Gamma(k_d))^{\langle\phi\rangle}$ when $(d,i) = 1$, it is sufficient to consider irreducible characters of the form $\theta_{\xi,\lambda} = \tilde{\xi}\lambda$, of dimension one, with $\xi \in X(\Gamma(k_d))^{\langle\phi\rangle}$. In this case $H_\xi = \langle\phi\rangle$ and $|H_\xi| = d$, so we have

$$(\tilde{\Lambda}_l - \Lambda'_l, \tilde{\xi}\lambda)_{\tilde{\Gamma}} = \sum\nolimits'_{\{i|\,\xi\in E_{l,i}\}} \frac{m_l(q)}{d} z_{l,i}\lambda^{-1}(\phi^i) - \sum\nolimits'_{\{i|\,\xi\in F_l^{\langle\phi\rangle}\}} \frac{m_l(q)}{d} z_{l,0}\lambda^{-1}(\phi^i).$$

If there exists $\xi \in F_l^{\langle\phi\rangle} \cap E_{l,i}$, then $z(\xi : k_{dl}|k_d) = z_{l,i} = z_{l,0}$. From this observation we easily obtain parts (ii) and (iii) of the Proposition. □

Proposition 2.6. *Theorem 2 for the case in which n is a prime number.*

Proof. In general, we want to show that $\tilde{\Lambda}_l = \Lambda'_l$ for all l, since this fact obviously implies Theorem 2. In the case n a prime number l assumes only the values 1 and n and we can quickly derive Theorem 2 by putting together (2.35) and (2.36). Propositions 2.2 and 2.4 and the fact that $\tilde{\Lambda}_1 = \Lambda'_1$ imply that $\tilde{\Lambda}_n - \Lambda'_n$ is a virtual character when n is prime. Therefore, for $(d,i) = 1$, we know that $(\tilde{\Lambda}_n - \Lambda'_n, \tilde{\xi})_{\tilde{\Gamma}}$ is an integer for all $\xi \in X(\Gamma(k_d))^{\langle\phi\rangle}$. But (2.35) and (2.36) imply that this is possible only if $E_{n,j} = F_n$ for all $(d,j) = 1$. Thus $\chi_{dn,j} \in [\chi_{dn}]$ for all $(d,j) = 1$. □

For $(d,i) = 1$ define the $\mathbb{Z}$-valued function

$$\varepsilon_\xi(l,i) = \begin{cases} z_{l,i}, & \text{if } \xi \in E_{l,i} - F_l \\ -z_{l,0}, & \text{if } \xi \in F_l^{\langle\phi\rangle} - E_{l,i} \\ 0, & \text{otherwise,} \end{cases} \tag{2.37}$$

and the $\mathbb{Q}$-valued sum

$$T_{\xi,i} = \sum_{l|n} \varepsilon_\xi(l,i)\frac{m_l(q)}{d}. \tag{2.38}$$

In terms of this new terminology Proposition 2.5 (ii), (iii) has the reformulation

$$\sum_{l|n}(\tilde{\Lambda}_l - \Lambda'_l, \tilde{\xi}\lambda)_{\tilde{\Gamma}} = {\sum_i}' T_{\xi,i}\lambda^{-1}(\phi^i) \tag{2.39}$$

for all $\xi \in X(\Gamma(k_d))^{\langle\phi\rangle}$ and $\lambda \in X(\langle\phi\rangle)$, where the sum ${\sum_i}'$ is over all i such that $(d,i) = 1$.

Proposition 2.7. *For all i such that $(d,i) = 1$ the sum $T_\xi := T_{\xi,i}$ is independent of i.*

Proof. Fixing ξ, we write $T_i := T_{\xi,i}$. Since $\sum_{l|n}(\tilde{\Lambda}_l - \Lambda'_l)$ is a virtual character, we know that ${\sum_i}' T_i\lambda^{-1}(\phi^i) \in \mathbb{Q}$ for all $\lambda \in X(\langle\phi\rangle)$ and all i. Let τ be a fixed primitive character of $\langle\phi\rangle$. Then $\lambda^{-1} = \tau^k$ for some k, $0 \le k < d$, and

$${\sum_i}' T_i\lambda^{-1}(\phi^i) = {\sum_i}' T_i\tau^k(\phi^i) = {\sum_i}' T_i\tau^i(\phi^k),$$

which is a rational number for all k and any fixed i such that $(d,i) = 1$. Thus $f(\phi^k) = {\sum_i}' T_i\tau^i(\phi^k)$ is a $\mathbb{Q}$-valued function on $\langle\phi\rangle$. Let $\mathfrak{G} := \mathrm{Gal}(\mathbb{Q}(\zeta_d)|\mathbb{Q})$, the Galois group of the d-th cyclotomic field $\mathbb{Q}(\zeta_d)|\mathbb{Q}$. For any function $h(\phi^k)$ on $\langle\phi\rangle$ with values in $\mathbb{Q}(\zeta_d)$ define

$$\sigma(h)(\phi^k) := \sigma(h(\phi^k))$$

for $\sigma \in \mathfrak{G}$. Clearly, f is a $\mathfrak{G}$-invariant function, so

$$f = \sigma(f) = {\sum_i}' T_i\sigma(\tau^i).$$

But $\mathfrak{G}$ acts transitively on the set $\{\tau^i | (d,i) = 1\}$ of primitive characters of $\langle\phi\rangle$ and this set is linearly independent over $\mathbb{C}$. Therefore the different presentations of f arising from $f \mapsto \sigma(f)$ can be the same function only if T_i is independent of i. $\square$

For $(d,i) = 1$ we set

$$\Delta_i := \bigcap_{\{l<n|\ l|n\}} E_{l,i}. \tag{2.40}$$

It is clear that $E_{l,i} \subset E_{l',i}$ for all factors $l'|l$. In particular, $E_{n,i} \subseteq \Delta_i$.

Proposition 2.8. *Assume that* $(d,i)=1$.

(i) If $\eta \in E_{n,i} - F_n$, *then*

$$T_{\eta,i} = \sum_{\{l|\ l|n,\, \eta\notin F_l\}} \frac{m_l(q)}{d} z_{l,i} = \frac{1}{d} + \sum_{\{l<n|\ l|n,\, \eta\notin F_l\}} \frac{m_l(q)}{d} z_{l,i}. \tag{2.41}$$

(ii) If $\eta \in \Delta_i - E_{n,i}$, *then*

$$T_{\eta,i} = \begin{cases} \sum_{\{l<n|\ l|n,\, \eta\notin F_l\}} \frac{m_l(q)}{d} z_{l,i}, & \text{if } \eta \notin F_n, \\ -\frac{1}{d} & \text{if } \eta \in F_n. \end{cases} \tag{2.42}$$

Proof. If $\eta \in E_{n,i}$, then $\eta \in E_{l,i}$ for all l. Therefore,

$$\varepsilon_\eta(l,i) = \begin{cases} z_{l,i}, & \text{for } \eta \notin F_l, \\ 0, & \text{for } \eta \in F_l. \end{cases}$$

This implies (i); for the second equality of (i) we note that $m_n(q)=1$ and $z_{n,i}=1$, since $\chi_{dn,i}$ is k_d-regular. If $\eta \in \Delta_i - E_{n,i}$, then

$$\varepsilon_\eta(l,i) = \begin{cases} z_{l,i}, & \text{for } \eta \notin F_l, \\ 0, & \text{for } \eta \in F_l \text{ with } l<n, \\ -1, & \text{for } \eta \in F_n. \end{cases}$$

Since $\eta \in F_n$ implies $\eta \in F_l$ for all $l|n$, it follows that $\varepsilon_\eta(l,i)=0$ for $l<n$. This proves (ii). □

Proposition 2.9. *Assume that* $\eta \in \Delta_i - F_n$ *for some* i *such that* $(d,i)=1$. *If, in addition,* $\eta \notin E_{n,i}$, *then* $\eta \notin E_{n,j}$ *for all* $(d,j)=1$.

Proof. Assume $\eta \in E_{n,j}$. By hypothesis, $\eta \notin F_n$, so, by Proposition 2.8 (i),

$$T_{\eta,j} = \frac{1}{d} + \sum_{\{l<n|\ l|n,\ \eta\notin F_l\}} \frac{m_l(q)}{d} z_{l,j}.$$

In this case, Proposition 2.8 (ii) implies that

$$T_{\eta,i} = \sum_{\{l<n|\ l|n,\, \eta\notin F_l\}} \frac{m_l(q)}{d} z_{l,i},$$

which contradicts $T_{\eta,i} = T_{\eta,j}$ (Proposition 2.7), since it is clear that $z_{l,i} = z_{l,j} = z(\eta : k_{dl}|k_d)$. □

Proposition 2.10. *If there exists i, $0 < i < d$, such that $(d,i)=1$ and such that $\Delta_i \not\subset E_{n,i} \cup F_n$, then Theorem 2 is true.*

Proof. Assume that there exists $\eta \in \Delta_i - (E_{n,i} \cup F_n)$. Then Proposition 2.9 implies that $\eta \notin \bigcup_{\{j|\ (d,j)=1\}} E_{n,j}$. By the induction hypothesis

$$E_{n,j} = F_n^{\langle \phi^j \rangle} \tag{2.43}$$

for all j such that $(d,j) > 1$. Thus,

$$\eta \notin E_{n,j} \tag{2.44}$$

for all j, $0 \leq j < d$. Since $\eta \in E_{l,i}$ for $l < n$ and every element of $E_{l,i}$ is $\langle \phi^i \rangle$-invariant, η is $\langle \phi \rangle$-invariant, since $(d,i)=1$. We may therefore consider the character $\tilde{\eta}$ of $\tilde{\Gamma}$ and combine (2.44) with Proposition 2.5 (ii), (iii) (the case $\lambda = 1$) with our assumption that $\eta \notin F_n$ to deduce that

$$(\tilde{\Lambda}_n - \Lambda'_n, \tilde{\eta})_{\tilde{\Gamma}} = 0. \tag{2.45}$$

Moreover, for $\eta \in \Delta_i$,

$$(\tilde{\Lambda}_l - \Lambda'_l, \tilde{\eta})_{\tilde{\Gamma}} = (\tilde{\Lambda}_l - \Lambda'_l, \tilde{\chi}_{dn,i})_{\tilde{\Gamma}} \tag{2.46}$$

for all $l < n$ such that $l|n$, since $\chi_{dn,i} \in \Delta_i$ too. By (2.45) and (2.46),

$$\Big(\sum_{l|n} (\tilde{\Lambda}_l - \Lambda'_l), \tilde{\chi}_{dn,i} - \tilde{\eta}\Big)_{\tilde{\Gamma}} = (\tilde{\Lambda}_n - \Lambda'_n, \tilde{\chi}_{dn,i})_{\tilde{\Gamma}}. \tag{2.47}$$

By Propositions 2.2 and 2.4, $\sum_{l|n}(\tilde{\Lambda}_l - \Lambda'_l)$ is a virtual character, which implies that the left side of (2.47) represents an integer. On the other hand, Proposition 2.5 (ii),(iii) for $l = n$ gives the right side as

$$(\tilde{\Lambda}_n - \Lambda'_n, \tilde{\chi}_{dn,i})_{\tilde{\Gamma}} = \begin{cases} \sum'_{\{j|\ \chi_{dn,i} \in E_{n,j}\}} \frac{1}{d}, & \text{if } \chi_{dn,i} \notin F_n, \\ -\sum'_{\{j|\ \chi_{dn,i} \notin E_{n,j}\}} \frac{1}{d}, & \text{if } \chi_{dn,i} \in F_n. \end{cases} \tag{2.48}$$

But (2.48) represents an integer if and only if $\chi_{dn,i} \in F_n \cap E_{n,j}$ for all $(d,j)=1$, which implies that the sum on the right is zero. Combining this with the induction hypothesis (2.43), we conclude that $\chi_{dn,i} \in F_n \cap E_{n,j}$ for all $j = 0, \dots, d-1$. $\square$

In order to complete the proof of Theorem 2 it is sufficient to show that there exists an i such that $(d,i)=1$ and such that $\Delta_i \not\subset E_{n,i} \cup F_n$. Since for $(d,i)=1$ we have $E_{l,i} \subset X(\Gamma(k_d))^{\langle \phi \rangle}$ for all $l|n$ (so $\Delta_i \subset X(\Gamma(k_d))^{\langle \phi \rangle}$), it suffices to show that $\Delta_i \not\subset E_{n,i} \cup F_n^{\langle \phi \rangle}$. Since each of these sets lies in $X(\Gamma(k_d))^{\langle \phi \rangle}$, we may pull these sets back via $\mathrm{N}_{k_{dn}|k_n}$ to subsets of $X(k_n^\times)$. Thus we send $E_{l,i}$ bijectively to

$$E^\circ_{l,i} := \{\eta' \in X(k_n^\times) |\ \eta'|_{k_l^\times} \underset{k}{\sim} \chi_{n,i}|k_l^\times\}, \tag{2.49}$$

$F_n^{\langle\phi\rangle}$ to

$$F_n^\circ := \{\eta' \in X(k_n^\times) \mid \eta' \underset{k}{\sim} \chi_n\} = [\chi_n], \tag{2.50}$$

and Δ_i to

$$\Delta_i^\circ := \bigcap_{\{l<n \mid\ l|n\}} E_{l,i}^\circ. \tag{2.51}$$

Let $C \subset k_n^\times$ be the subgroup generated by the set of all multiplicative groups $\{k_l^\times \mid l < n,\, l|n\}$. Let $Y := X(k_n^\times/C)$, the subgroup of $X(k_n^\times)$ consisting of all characters which are trivial on the multiplicative groups of all proper subfields, and let $S_{n,i} := \bigcup_{\eta'\in[\chi_{n,i}]} \eta' Y$. If $\Delta_i \subset E_{n,i} \cup F_n^{\langle\phi\rangle}$, then

$$[\chi_{n,i}] = E_{n,i}^\circ \subseteq S_{n,i} \subseteq \Delta_i^\circ \subseteq E_{n,i}^\circ \cup E_{n,0}^\circ. \tag{2.52}$$

Each of the sets in (2.52) is stabilized by $\mathrm{Gal}(k_n|k)$, so each consists either of a single $\mathrm{Gal}(k_n|k)$-orbit or of the union of two $\mathrm{Gal}(k_n|k)$-orbits of regular characters. Therefore, writing $[\chi_{n,i}|_C]$ for the $\mathrm{Gal}(k_n|k)$-orbit of the restriction $\chi_{n,i}|_C$ and noting that multiplication by Y stabilizes $S_{n,i}$, we see that

$$|S_{n,i}| = |[\chi_{n,i}|_C]| \cdot |Y| \in \{n, 2n\}. \tag{2.53}$$

For $(n,q) \neq (6,2)$ or $(2,3)$ the equality of (2.53) is impossible, because, by [SZ, Lemma 1.2], $\Phi_n(q)$ divides $|Y|$ ($\Phi_n(X)$ denotes the n-th cyclotomic polynomial) and, according to [ZS] or [AR, §1, Corollary 2], for all $(n,q) \neq (6,2)$, $(2,3)$ we have $\ell \mid \Phi_n(q)$, where $\ell > n$ is a prime. This proves:

Proposition 2.11. *For all $(n,q) \neq (6,2)$ or $(2,3)$ there exists i such that $(d,i) = 1$ and $\Delta_i \not\subset E_{n,i} \cup F_n$.*

Propositions 2.10 and 2.11 imply Theorem 2 for all $(n,q) \neq (6,2), (2,3)$. In Proposition 2.6 we proved Theorem 2 for all prime n; in particular this includes the case $(n,q) = (2,3)$.

We have only to complete the proof of Theorem 2 in the case $(n,q) = (6,2)$. In this case, $|Y| = [\mathbb{F}_{2^6}^\times : \mathbb{F}_{2^3}^\times \cdot \mathbb{F}_{2^2}^\times] = 3$. From now on let $k := \mathbb{F}_2$. In the present instance, if $\Delta_i^\circ \subset E_{6,i}^\circ \cup F_6^\circ$, then the set $S_{6,i}$ is comprised only of regular characters and is $\mathrm{Gal}(k_6|k)$-stable. From (2.53) it follows that $|S_{6,i}| \in \{6, 12\}$ and from the fact that the Galois orbit of the restriction to C satisfies $|[\chi_{6,i}|_C]| \in \{2,3,6\}$ it follows that $|[\chi_{6,i}] \cdot Y| \in \{6, 9, 18\}$. Therefore,

$$|S_{6,i}| = 6, \qquad S_{6,i} = E_{6,i}, \qquad \text{and} \qquad |[\chi_{6,i}|_C]| = 2. \tag{2.54}$$

To see that there is only one orbit of regular characters in $X(k_6^\times)$ which satisfies the three conditions (2.54), we identify the cyclic group $X(k_6^\times)$ with $(\mathbb{Z}/63)^+$ and we let χ_1 correspond to the generator $1 \in (\mathbb{Z}/63)^+$. Then Y corresponds to the set of multiples of 21. Represented as a subset of $(\mathbb{Z}/63)^+$, a regular Galois orbit has the form $[a] = \{a, 2a, \dots 2^5 a\} \subset (\mathbb{Z}/63)^+$. Since $Y \cdot [\chi_{n,i}] = [\chi_{n,i}]$, it follows that $21 + [a] = [a] \subset (\mathbb{Z}/63)^+$ too. Clearly, $[a] = [7]$ corresponds to the only Y-stable Galois orbit of regular characters. Expressed otherwise, $[\chi_1^7]$ is the only Galois orbit of regular characters which is Y-stable.

We have proved:

Lemma 2.12. *Assume that* $(n, q) = (6, 2)$. *If* $\Delta_i \subset E_{6,i} \cup F_6$ *for all* $(d, i) = 1$, *then* $E^\circ_{6,i} = [\chi_1^7]$.

Let us elaborate Lemma 2.12 to show that it completes the proof of Theorem 2. Fix any i such that $(d, i) = 1$. Then Shintani's Theorem and Theorem 1 imply that the descent mapping j_i defines a bijective mapping from the set of $\langle\phi\rangle$-invariant cuspidal representations of $\mathrm{G}_6(k_d)$ to the set of all cuspidal representations of $\mathrm{G}_6(k)$. Representing cuspidal representations of both $\mathrm{G}_6(k_d)$ and $\mathrm{G}_6(k)$ by their Green's parameters and using Theorem 1 to identify the set of $\langle\phi\rangle$-invariant $\mathrm{Gal}(k_{d6}|k_d)$-orbits of regular characters in $X(k_{d6}^\times)$ with the set of $\mathrm{Gal}(k_6|k)$-orbits of regular characters in $X(k_6^\times)$, we regard j_i as a permutation of the set of Green's parameters of the cuspidal representations of $\mathrm{G}_6(k)$. In this context, if $j_i([\chi]) \neq [\chi_1^7]$, then, by Lemma 2.12, $\Delta_i \not\subset E_{6,i} \cup F_6$ and, by Proposition 2.10, $j_i([\chi]) = [\chi]$ has to be fixed by the permutation j_i. Thus only the orbit $[\chi_1^7]$ can fail to be fixed by j_i. Since a permutation which acts as the identity at all but one element of a set has to be the identity permutation, j_i acts as the identity at $[\chi_1^7]$ too. We conclude that Theorem 2 is true in the case $(n, q) = (6, 2)$ too.

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